Conformally covariant differential operators acting on spinor bundles and related conformal covariants

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To my family,
which always believed in me.

Abstract

Conformal powers of the Dirac operator on semi Riemannian spin manifolds are investigated. We give a new proof of the existence of conformal odd powers of the Dirac operator on semi Riemannian spin manifolds using the tractor machinery. We will also present a new family of conformally covariant linear differential operators on the standard spin tractor bundle. Furthermore, we generalize the existence proof of conformal power of the Dirac operator on Riemannian spin manifolds [GMP12] to semi Riemannian spin manifolds. Both proofs concering the existence of conformal odd powers of the Dirac operator are constructive, hence we also derive an explicit formula for a conformal third- and fifth power of the Dirac operator. Due to explicit formulas, we show that the conformal third- and fifth power of the Dirac operator is formally self-adjoint (anti self-adjoint), with respect to the L^2-scalar product on the spinor bundle. Finally, we present a new structure of the conformal first-, third- and fifth power of the Dirac operator: There exist linear differential operators on the spinor bundle of order less or equal one, such that the conformal first-, third- and fifth power of the Dirac operator is a polynomial in these operators.

Zusammenfassung

Konforme Potenzen des Dirac Operators einer semi Riemannschen Spin-Mannigfaltigkeit werden untersucht. Wir präsentieren einen neuen Beweis, basierend auf dem Tractor Kalkül, für die Existenz von konformen ungeraden Potenzen des Dirac Operators auf semi Riemannschen Mannigfaltigkeiten. Desweiteren konstruieren wir eine neue Familie von konform kovarianten linearen Differentialoperatoren auf dem standard spin Traktor Bündel. Weiterhin verallgemeinern wir den Existenzbeweis für konforme ungerade Potenzen des Dirac Operators, präsentiert in [GMP12], auf semi Riemannsche spin Mannigfaltigkeiten. Da die Existenzbeweise konstruktive sind, erhalten wir explizite Formeln für die konforme dritte und fünfte Potenz des Dirac Operators. Basierend auf den expliziten Formeln zeigen wir, dass die konforme dritte und fünfte Potenz des Dirac Operators formal selbstadjungiert (anti selbstadjungiert) bezüglich des L^2-Skalarproduktes auf dem Spinorbündel ist. Abschliessend präsentieren wir neue Strukturen der konformen ersten, dritten und fünften Potenz des Dirac Operators: Es existieren lineare Differentialoperatoren auf dem Spinorbündel der Ordnung kleiner gleich eins, so dass die konforme erste, dritte und fünfte Potenz des Dirac Operators ein Polynom in jenen Operatoren ist.

Contents

1 Introduction

The study of conformally covariant linear differential operators was stimulated by mathematicans and physicists at the early 20th century. From the mathematical point of view, conformal structures are generalizations of Bernhard Riemann's idea to describe a geometry through a manifold equipped with a metric. Hermann Weyl, a pioneer of conformal geometry tried to unify the theory of general relativity with that of electrodynamics [Wey18] within the framework of conformal geometry. One application of conformal geometry is that statements within Riemannian geometry which do not hold for a specific metric can become true after a conformal change of this metric, e.g. as in the Yamabe problem [Yam60, LP87]. Conformally covariant differential operators acting on vector bundles over a semi Riemannian manifold are a special class of linear geometrical differential operators on these vector bundles. From the spectral theoretical point of view, functional determinants of self-adjoint, elliptic and conformally covariant differential operators are conformal invariants on odd dimensional manifolds [PR87, Ros87]. From the physical point of view, conformal transformations preserve the light-cone, hence the path of massless particles. In flat space Maxwell's equations and the massless Dirac equation are invariant under the conformal group [Bat10, Haa41]. Thus, these field theories are invariant under the conformal group. In modern physics, a quantum field theory (QFT) which yields a invariance not only under the Poincaré group but also under the conformal group is referred to a conformal quantum field theory (CFT). Functional determinants of linear operators also play an important role in quantum field theory, where they are used for the definition of the path integral. The AdS/CFT-correspondence proposed by Maldacena [Mal99] is a relation between gravitational theories modelled on a space and conformal quantum theories without gravity modelled on its conformal boundary.

A special class of geometric linear differential operators acting between vector bundles over a semi Riemannian manifold is given by those, transforming nicely under a conformal change of the semi Riemannian metric, i.e., which are conformally covariant differential operators of a certain bi-degree in the sence of [Kos75]. The Dirac operator on a spin manifold is conformally covariant [Hit74], whereas for the Laplacian acting on functions this is not true in general. One has to modify the Laplacian by a multiple of the scalar curvature [Yam60, Ørs76] in order to obtain the conformal covariance. A systematic approach into the direction of a classification of conformally covariant differential operators was done in [Feg76] and [Bra98]; Fegan classified all first order conformally covariant differential operators, acting between vector bundles equipped with a Riemannian structure, and Branson did the same for second order differential operators. Higher order conformally covariant differential operators were rarely considered. The Yamabe operator acting on functions was generalized by Branson [Bra82] to differential $k-$forms.

Just one year later, Paneitz [Pan08] constructed a fourth order conformally covariant operator acting on functions, which was generalized to differential k-forms by Branson in [Bra85], where he also constructed a sixth order conformally covariant differential operator on functions. Following the time line, Branson and Kosmann-Schwarzbach [BK83] have shown that each conformally covariant differential operator on spinors can be modified by a term, nonlinear in the spinor, without losing the covariance property. In [Bra84], Branson introduced first order differential operators on spinor $k-$forms which generalize the Dirac operator. At the end of the eighties, Jenne [Jen88] used the ambient metric construction [FG85, FG07, FG11], a powerful tool in conformal geometry, to derive conformally covariant differential operators that act on tensor fields, with various symmetries, and that have the iterated divergence, the Laplacian or some symmetrized covariant derivatives as leading terms, respectively. In the 1990's, the authors Graham, Jenne, Mason and Sparling [GJMS92] constructed a series of conformally covariant differential operators with leading term some power of the Laplacian, called GJMS-operators, which generalised the Yamabe and Paneitz operator. At the beginning of this century, Holland and Sparling [HS01], inspired by the techniques of [GJMS92], proved the existence of conformally covariant differential operators acting on the spinor bundle, with leading term some odd power of the Dirac operator. Beside the general existence of conformal powers of the Dirac operator, Branson [Bra05] has derived a conformal third power of the Dirac operator in the framework of tractor calculus. In the same year, Branson and Gover [BG05] have introduced the so called Branson-Gover-operators, which are conformally covariant differential operators on differential $k-$forms which have as leading term a sum of powers of compositions of exterior differential and co-differential and vice versa. From the spectral theoretical point of view, Graham and Zworski [GZ03] have shown that the GJMS-operators on a Riemannian manifold are also encoded in the residues of the scattering matrix of the Laplacian of the associated Poincaré model. The case of conformal powers of the Dirac operator on Riemannian manifolds was handled by Guillarmou, Moroianu and Park [GMP10] and [GMP12], where conformal powers of the Dirac operator appear in the scattering operator of the Dirac operator of the Poincaré model. Additionally, they gave an explicit formula for a third conformal power of the Dirac operator, which agrees with the conformal third power of the Dirac operator given by Branson [Bra05]. Juhl's study of $Q-$curvature in [FJ11, Juh09b] leads to a fundamental structure of the GJMS-operators [Juh10, Juh11], which was reproved by Fefferman and Graham in [FG13].

Besides that development of conformally covariant differential operators on general curved manifolds many special cases were considered. In the case of homogeneous vector bundles, Slovak [Slo93, Chapter 8] classified all conformally covariant operators using the technique of Verma modules. In particular, in the spinorial case no even order conformally covariant differential operators with the same source and target space exist. In case of Minkowski space, Jakobson and Vergne [JV77] have proved that powers of the Laplacian and odd powers of the Dirac operator are intertwining operators between representations of principal series, hence, that they are conformally equivariant. Eastwood and Rice [ER87] have described a construction of conformally covariant differential operators on Minkowski space via the translation principle and have discussed their generalizations to the curved case. On the sphere, Branson [Bra95] derived an explicit formula for conformal powers of the Laplacian as products of shifted Laplacians associated to the round metric on the sphere. More general, Gover [Gov06] has shown, that on conformally Einstein manifolds the GJMS-operators obey a product structure in terms of the Laplacian associated to the Einstein metric inside the conformal class. Analogously, the case of conformal powers of the Dirac operator were only worked out for the sphere by Branson and Ørsted [BØ06], and Eelbode and Souček [ES10]. More recently, in the setting of spinor $k-$forms on the standard sphere, Hong [Hon11] introduced conformally covariant differential operators with leading term some combination of the differential and co-differential, like in the case of differential $k-$forms. Finally, Šmíd [Š12] found a fifth order conformally invariant higher spin operator on the sphere.

The aim of the thesis is the investigation of conformal powers of the Dirac operator $\not{D}$ on an $n-$dimensional semi Riemannian spin manifold (M, g), i.e., conformally covariant linear differential operators acting on the spinor bundle with leading term an odd power of the Dirac operator. As mentioned before, there are no conformal powers of the Dirac operator of even order. Beside the existence of conformal powers of the Dirac operator, there is no concept of uniquness, since one can add to them lower order conformally covariant differential operators. We will present several constructions of conformal powers of the Dirac operator. These constructions are algorithmic, thus we are able to present an explicit formula for a conformal third- and fifth powers of the Dirac operator. Due to these explicit formulas, we prove, that the conformal third- and fifth power of the Dirac operator is depending on the signature formally self-adjoint (anti self-adjoint) with respect to the L^2-scalar product on the spinor bundle.

Now, let us summarize the results of this thesis. We present a new proof of the existence of conformal powers of the Dirac operator on spin manifolds using the tractor machinery, see Theorem 5.27. Furthermore, we generalize the result, obtained in [GMP12], concerning the existence of conformal powers of the Dirac operator, to semi Riemannian manifolds, see Theorem 4.17. The well known conformal third power of the Dirac operator, derived in a new way in Theorem 5.48, and the conformal fifth power of the Dirac operator, derived in Theorem 5.39 and in Theorem 5.49, are given by

$$\begin{aligned}
\mathcal{D}_3 =& \not{D}^3 - (P, \nabla^{S(M,g)}) - (\nabla^{S(M,g)}, P\cdot),\\
\mathcal{D}_5 =& \not{D}\mathcal{D}_3\not{D} + 2[\not{D}^2\mathcal{D}_3 + \mathcal{D}_3\not{D}^2] - 4\not{D}^5 + 8\big[(P^2, \nabla^{S(M,g)}) + (\nabla^{S(M,g)}, P^2\cdot)\big]\\
&+ \frac{4}{n-4}\big[(B, \nabla^{S(M,g)}) + (\nabla^{S(M,g)}, B\cdot)\big] - 2\big[(C, P\cdot) + (P, C\cdot)\big],
\end{aligned}$$

where P, C and B denote the Schouten, Cotton and Bach tensor, respectively. Required notation will be introduced in the equations (2.4), (2.5), (5.13) and (5.14). Both operators are formally self-adjoint (anti self-adjoint) with respect to the L^2-scalar product on the spinor bundle, see Theorem 5.41.

On the standard spin tractor bundle $\mathcal{S}(M)$ associated to a conformal spin manifold $(M, [g])$, there exist for any conformally covariant differential operators $D_k(g)$ on the spinor bundle of bi-degree $\left(\frac{k-n}{2}, -\frac{k+n}{2}\right)$, where $k \in \mathbb{N}$ is odd, a conformally covariant differential operator $L_k(g) : \Gamma(\mathcal{S}(M)) \to \Gamma(\mathcal{S}(M))$ of bi-degree $\left(\frac{k+1-n}{2}, -\frac{k+1+n}{2}\right)$, see Theorem 5.46.

The last result presented in Chapter 6 yields some structural insight into the first examples of conformal powers of the Dirac operator. Namely, there exist differential operators M_1, M_3 and M_5, of order less or equal one, acting on the spinor bundle, which are not conformally covariant in general, and which are the building blocks for $\mathcal{D}_1 = \not{D}$, $\mathcal{D}_3$ and $\mathcal{D}_5$, see Theorem 6.1.

This thesis is organized as follows: Chapter 2 provides the basic materials concerning differential geometry. In order to fix some notation, we will briefly review concepts of fibre bundles, especially principal and vector bundles, and the theory of connections and covariant derivatives. Also special structures on manifolds, like Riemannian-, spin- and Cartan structures, will be discussed. For more details about these subjects see [KN63, Bau09, LM89, Sha97].

Chapter 3 will introduce the reader to conformal geometry, the study of angle preserving structures. We will present how a conformal change of a semi Riemannian metric will influence on standard objects of semi Riemannian and spin geometry. One approach to conformal geometry is given within the framework of Cartan geometries. The Möbius sphere, the standard flat model of conformal geometry, naturally carries the structure of a Cartan geometry. This will be generalized in terms of the first prolongation of the conformal frame bundle associated to a general conformal manifold. A feature of the first prolongation is the existence of a distinguished Cartan connection [ČSS97a, ČSS97b, Feh05]. These structures replace the orthonormal frame bundle together with the Levi-Civita connection for semi Riemannian manifolds, if one is dealing with conformal structures. This phenomenon is part of a more general theory, that of parabolic geometry [ČS09]. By analogy with spin structures equipped with the spin connection associated to a semi Riemannian structure, we present the first prolongation of a conformal spin structures together with a Cartan connection induced by the distinguished Cartan connection of the first prolongation associated to a conformal structure. Another way to derive a suitable calculus for conformal geometry is due to the ambient metric construction of Fefferman and Graham [FG85, FG07, FG11], which we will recall. The latter two ways of looking at conformal geometry are known to be equivalent [ČG03]. Finally, we define (infinitessimal) conformally covariant differential operators and show that both definitions are equivalent. From the computational point of view, the infinitesimal version of conformal covariance is more practicable, since only linear terms of the conformal factor are involved in the computations.

Chapter 4 deals with a construction of conformal powers of the Dirac operator from a spectral theoretical point of view. For Riemannian spin manifolds, it was shown in [GMP10, GMP12], that the scattering operator of the Dirac operator of the Poincaré model contains conformal powers of the Dirac operator associated to the boundary, and an explicit formula for a conformal third power of the Dirac operator was given, in agreement with the result obtained by Branson [Bra05]. We will carry out this construction in a rather elementary way, and we do not restrict ourselves to Riemannian spin manifolds. Working within the Poincaré model of a conformal manifold we formally solve the eigenequation with respect to the Poincaré Dirac operator through a well chosen ansatz. This requires a suitable calculus for hypersurfaces and their associated spinor bundles, which we will recall, see [Bur93, BGM05]. For special eigenvalues $\lambda = -\frac{1}{2}, -\frac{3}{2}, \ldots$, the formal solution of the eigenequation will contain a conformal power of the Dirac operator. In this setting we will derive an explicit formula for a conformal third power of the Dirac operator, using the explicit knowledge of the first terms of the formal power series of the $1-$parameter family of metrics induced by the Poincaré metric and the concept of a first variation of the Dirac operator, see [BG92, BGM05], applyed to that $1-$parameter family of metrics. A very similar computation without explicit use of the first variation of the Dirac operator was done in [GMP12].

Chapter 5 presents a new construction of conformal powers of the Dirac operator via the tractor calculus. We will derive explicit formulas for a conformal third- and fifth power of the Dirac operator. Again, the obtained conformal third power of the Dirac operator agrees with the one found by Branson [Bra05], even though the construction differs. The philosophy of tractor bundles and their tools, i.e., tractor connection, tractor D-operator, tractor scalar product, etc., is well presented in [BEG94]. A more fundamental development of structures behind tractor bundles and their tools is given in [ČG00, ČG01, ČS09]. Our construction of conformal powers of the Dirac operator is based on the curved translation principle introduced by Eastwood and Rice [ER87] and is similar to the construction of conformal powers of the Laplacian given in [GP03]. First we will construct a family $P_{2N}^{\mathcal{S}(M)}(g)$ of conformally covariant differential operators acting on the standard spin tractor bundle by composing the Box-operator $\Box_{g,\frac{2-n}{2}}^{\mathcal{S}(M)}$ several times with appropriate tractor D-operators from the right and tractor C-operators from the left. A further composition of the family $P_{2N}^{\mathcal{S}(M)}(g)$ with the tractor D-operator of the spinor bundle from the right and with the tractor C-operator of the spinor bundle from the left induces a family of conformally covariant differential operators acting on the spinor bundle, which are conformal powers of the Dirac operator. Since we are interested in explicit formulas for conformal powers of the Dirac operator up to order five, it is absolutely necessary to have a precise knowledge of the $g-$metric representation of corresponding tractor objects. We have to point out, that the conformal fifth power of the Dirac operator obtained by this construction decomposes into two conformally covariant differential operators. One of these two operators is of order less than two. This is very similar to the construction of a conformal second power of the Laplacian via the tractor calculus, see [GP03]. Due to explicit formulas for a conformal third- and fifth power of the Dirac operator we prove that they are formally self-adjoint (anti self-adjoint) with respect to the L^2-scalar product on the spinor bundle. Finally, we will present a new method to obtain a conformally covariant differential operator on the standard spin tractor bundle, induced by a conformally covariant differential operator on the spinor bundle: Starting with a conformally covariant differential operator $D_k(g)$ on the spinor bundle of bi-degree $\left(\frac{k-n}{2}, \frac{k+n}{2}\right)$, where $k \in \mathbb{N}$ is odd, we compose it with the tractor D-operator of the spinor bundle from the left and with the tractor C-operator of the spinor bundle from the right. This gives us a conformally covariant differential operator $L_k(g)$ on the standard spin tractor bundle. It is remarkable that a conformal power of the Dirac operator induced by $P_{2N}^{\mathcal{S}(M)}(g)$, for $N \in \mathbb{N}$, will induce $L_{2N-1}(g)$, which is different from $P_{2N}^{\mathcal{S}(M)}(g)$.

Chapter 6 is the final chapter and collects some further results. Mainly, it presents some new insights into the structure of the conformal first-, third- and fifth power of the Dirac operator: There exist differential operators of order less than two, such that the conformal first-, third- and fifth power of the Dirac operator is a polynomial in these operators. This is closely related to Juhl's inversion formulas [Juh11] for the conformal powers of the Laplacian acting on functions.

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2 Differential geometric background

This chapter reviews some basic concepts concerning differential geometry. The main goal is to fix notation and to collect formulas which will be useful later. For more details, we refer to standard literature about fibre bundles [Bau09] and [KN63], about spin geometry [Bau81] and [LM89], and about Cartan geometry [Sha97] and [ČS09].

The basic object in differential geometry is that of an $n-$dimensional manifold M. We will always assume that our manifolds in question are smooth, i.e., equipped with a $\mathcal{C}^\infty-$structure. A fibre bundle $\mathcal{F}$ over M is denoted by a quadruple $(\mathcal{F}, \pi, M, F)$, where the total space $\mathcal{F}$ is locally trivial, $\pi : \mathcal{F} \to M$ is the smooth projection, and F is the fibre of $\mathcal{F}$. There is a natural subspace $Tv\mathcal{F}_p := \ker(d\pi_p) \subset T_p\mathcal{F}$, called the vertical tangent space at a point $p \in \mathcal{F}$. The space of sections of a fibre bundle $(\mathcal{F}, \pi, M, F)$ is denoted by $\Gamma(\mathcal{F}) := \{\sigma \mid \sigma \in \mathcal{C}^\infty(M, \mathcal{F}), \pi \circ \sigma = \mathrm{id}_M\}$. Special cases of fibre bundles are principal bundles (P, π, M, G) and vector bundles (E, π, M, V) over M. A connection form $Z \in \Omega^1(P, \mathfrak{g})$ on a principal bundle (P, π, M, G) yields a decomposition $TP = ThP \oplus TvP$, where the horizontal distribution ThP is the kernel of Z. Let us consider a principal bundle (P, π, M, G) equipped with a connection form Z. Any representation $\rho : G \to Gl(V)$ of the Lie group G on some finite-dimensional vector space V induces a covariant derivative $\nabla^{E,Z}$ on the associated vector bundle $E := P \times_{(G,\rho)} V$, given locally by

$$(\nabla^{E,Z} e)_{|U} = [p, dv + \rho_*(Z^p(\cdot))v], \tag{2.1}$$

where $U \subset M$ is some open neighbourhood, $e = [p, v] \in \Gamma(E_{|U})$ for a local section $p : U \to P$ and for a smooth map $v : U \to V$, $\rho_* := d\rho_{1_G}$ is the differential of ρ at the identity $1_G \in G$, and $Z^p(\cdot) := Z \circ dp$ is the local connection form of Z with respect to the local section p. The curvature $\Omega^Z := dZ \circ \mathrm{proj}_{ThP} \in \Omega^2(P, \mathfrak{g})$ of Z is related to the curvature

$$R^{E,Z}(X,Y) := \nabla^{E,Z}_X \nabla^{E,Z}_Y - \nabla^{E,Z}_Y \nabla^{E,Z}_X - \nabla^{E,Z}_{[X,Y]},$$

of $\nabla^{E,Z}$ by $R^{E,Z}(X,Y)e = [p] \circ \rho_*(\Omega^Z(X^*, Y^*)) \circ [p]^{-1}$ for a local section $e = [p, v] : U \to E$, where $[p] : E \to V$ is the canonical isomorphism $e = [p, v] \mapsto v$, $X, Y \in \mathfrak{X}(M)$ and $X^*, Y^* \in \mathfrak{X}(P)$ are their horizontal lifts with respect to Z.

Let $(E_i, \pi_i, M, V_i, \nabla^i)$, $i = 1, 2$, be two vector bundles over M equipped with a covariant derivative. There are several ways of defining new vector bundles by forming Whitney sums, tensor products, dual and homomorphism bundles, where their covariant derivatives are $\nabla^{\oplus}_X(e_1 \oplus e_2) := \nabla^1_X e_1 \oplus \nabla^2_X e_2$, $\nabla^{\otimes}_X(e_1 \otimes e_2) := \nabla^1_X e_1 \otimes e_2 + e_1 \otimes \nabla^2_X e_2$, $(\nabla^*_X \eta_1)(e_1) = X(\eta_1(e_1)) - \eta_1(\nabla_X e_1)$ and $(\nabla^{hom}_X A)(e_1) := \nabla^2_X(A(e_1)) - A(\nabla^1_X e_1)$ for

$e_i \in \Gamma(E_i)$, $i = 1, 2$, $X \in \Gamma(TM)$, $\eta_1 \in \Gamma(E_1^*)$ and $A \in \Gamma\big(Hom(E_1, E_2)\big)$.

Consider a vector bundle (E, π, M, V) over M equipped with a covariant derivative ∇^E. Denoting the space of $k-$forms with values in E by $\Omega^p(M, E) := \Gamma(\Lambda^p M \otimes E)$ we define the exterior differential $d^{\nabla^E} : \Omega^p(M, E) \to \Omega^{p+1}(M, E)$ by

$$d^{\nabla^E} w(X_0, \ldots, X_p) := \sum_{i=0}^{p} (-1)^i \nabla^E_{X_i} w(X_0, \ldots, \hat{X}_i, \ldots, X_p)$$

where $w \in \Omega^p(M, E)$, $X_0, \ldots X_p \in \Gamma(TM)$ and $\hat{\ }$ denotes omitting the entry. In order to understand the geometry of a manifold, it is reasonable to consider several additional structures on the given manifold.

Semi Riemannian geometry: Let M be equipped with a semi Riemannian metric, i.e., a symmetric non-degenerate bilinear form $g \in \Gamma(T^*M \otimes T^*M)$ of signature (p, q), where p stands for the number of -1's arising from Sylvester's Theorem. We call such a pair (M, g) a semi Riemannian manifold of signature (p, q). A choice of a semi Riemannian metric on a manifold M is equivalent to an $O(p, q)-$reduction

$$\mathcal{P}^g := \left\{ s = (s_1, \ldots, s_n)_x \,\middle|\, \begin{array}{c} x \in M, s \text{ is an orthonormal basis in} \\ T_x M \text{ with respect to } g \end{array} \right\}$$

of the frame bundle $\big(GL(M), \pi, M, Gl(n, \mathbb{R})\big)$ to the structure group $O(p, q) \subset Gl(n, \mathbb{R})$. Therefore, the tangent bundle of a semi Riemannian manifold obeys the following isomorphisms

$$TM \simeq GL(M) \times_{(Gl(n,\mathbb{R}),\rho)} \mathbb{R}^n \simeq \mathcal{P}^g \times_{(O(p,q),\rho)} \mathbb{R}^n,$$

where ρ denotes the standard representation of $Gl(n, \mathbb{R})$ on $\mathbb{R}^n$. The Levi-Civita connection $\nabla^{LC} : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ induced by g determines a connection form $A^g \in \Omega^1(\mathcal{P}^g, \mathfrak{o}(p, q))$ which local connection $1-$form with respect to a local section $s = (s_1, \ldots, s_n) : U \to \mathcal{P}^g$ is given by

$$(A^g)^s(X) = \sum_{i,j=1}^{n} \varepsilon_j g(\nabla^{LC}_X s_i, s_j) B_{ij} = \sum_{1 \le i < j \le n} \varepsilon_i \varepsilon_j g(\nabla^{LC}_X s_i, s_j) E_{ij},$$

where $\varepsilon_i := g(s_i, s_i)$ for all $1 \le i \le n$, $\{B_{ij}\}_{i,j=1}^n$ denotes that basis of $\mathfrak{gl}(n, \mathbb{R})$ given in terms of a basis $\{e_i\}$ of $\mathbb{R}^n$ by $B_{ij} := e_j e_i^t$ and $\{E_{ij} := \varepsilon_i B_{ij} - \varepsilon_j B_{ji}\}_{i<j}^n$ is a basis for $\mathfrak{o}(p, q)$. The exterior derivative d on $p-$forms on M can be expressed in terms of the Levi-Civita connection by

$$dw(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i (\nabla^{LC}_{X_i} w)(X_0, \ldots, \hat{X}_i, \ldots, X_p)$$

for $w \in \Omega^p(M)$. Introducing the L^2-scalar product on compactly supported $p-$forms the formal adjoint of the exterior derivative with respect to this scalar product is the co-differential $\delta : \Omega^{p+1}(M) \to \Omega^p(M)$, which is given locally by

$$(\delta w)(X_1, \ldots, X_p) = -\sum_{i=1}^{p} \varepsilon_i (\nabla^{LC}_{s_i} w)(s_i, X_1, \ldots, X_p),$$

where $w \in \Omega^p(M)$ and $s = (s_1, \ldots, s_n)$ is a local section of $\mathcal{P}^g$. Note that δ depends on the semi Riemannian structure g.

Remark 2.1 Let $(E, \pi, M, V; h, \nabla)$ be a vector bundle over a semi Riemannian manifold (M, g) equipped with a bundle metric $h \in \Gamma(S^2E^*)$ and an $h-$metric covariant derivative ∇^E. The above notation of the co-differential can be extended to $p-$forms with values in E by considering the appropriate covariant derivative on $\Omega^p(M, E)$. It will be denoted by δ^{∇^E}. Note, that the definition of δ^{∇^E} do not require the bundle metric h. But, with respect to the bundle metric h the co-differential δ^{∇^E} is the formal adjoint of d^{∇^E} with respect to the L^2-scalar product on differential forms with values in E. From d^{∇^E} and δ^{∇^E} we obtain two second order differential operator acting on $p-$form with value in E:

$$\Delta_g^{\nabla^E} := \mathrm{tr}_g(\nabla^{T^*M \otimes E} \circ \nabla^E),$$
$$\Delta_p := d^{\nabla^E} \circ \delta^{\nabla^E} + \delta^{\nabla^E} \circ d^{\nabla^E}.$$

These are the Bochner-Laplacian and the Hodge-Laplacian of the bundle E, both are metric dependend.

Finally, let us mention some natural isomorphisms, tensors and formulas related to a semi Riemannian manifolds (M, g). The non-degeneracy of the metric g allows us to identify the tangent bundle with its dual: For $x \in M$ and $X \in T_xM$ let us denote by $X^\flat(\cdot) := g_x(X, \cdot)$ the dual vector corresponding to X. Vice versa, a dual vector $\eta \in T_x^*M$ induces a vector $\eta^\natural \in T_xM$ such that, $g_x(\eta^\natural, Y) = \eta(Y)$ for all $Y \in T_xM$.

For $X, Y, Z, W \in \Gamma(TM)$, we may define, starting from the curvature

$$R(X, Y) = \nabla^{LC}_X \nabla^{LC}_Y - \nabla^{LC}_Y \nabla^{LC}_X - \nabla^{LC}_{[X,Y]}$$

of the Levi-Civita connection, the standard tensors fields. These are

- the Riemann curvature tensor $\mathcal{R}(X, Y, Z, W) := g(R(X, Y)Z, W)$,
- the Ricci tensor $Ric(X, Y) := Tr_g\big(g(R(X, \cdot)\cdot, Y)\big)$,
- the scalar curvature $\tau := Tr_g\big(Ric(\cdot, \cdot)\big)$,
- the normalized scalar curvature $J := \frac{1}{2(n-1)}\tau$,
- the Schouten tensor $P(X, Y) := \frac{1}{n-2}\big(Ric(X, Y) - \frac{1}{2(n-1)}\tau g(X, Y)\big)$,

- the Weyl tensor $W(X,Y,Z,W) := \mathcal{R}(X,Y,Z,W) + (P \owedge g)(X,Y,Z,W)$,
- the Cotton tensor $C(X,Y,Z) := (\nabla_X P)(Y,Z) - (\nabla_Y P)(X,Z)$ and
- the Bach tensor $B(X,Y) = Tr_g((\nabla.C)(\cdot,X,Y)) + g\,(P, W(\cdot,X,Y,\cdot))$.

In the definition of the Weyl tensor, the Kulkarni-Nomizu product $\owedge$ of two symmetric $(0,2)-$tensors S_1 and S_2 appears, which is defined as follows:

$$\begin{aligned}(S_1 \owedge S_2)(X,Y,Z,W) :=& S_1(X,Z)S_2(Y,W) + S_1(Y,W)S_2(X,Z)\\ &- S_1(X,W)S_2(Y,Z) - S_1(Y,Z)S_2(X,W).\end{aligned}$$

Note that we sometimes consider a $(0,2)-$tensor as a $1-$form with values in the dual tangent bundle.

Remark 2.2 Consider a semi Riemannian manifold (M,g), a point $x \in M$ and a normal neighbourhood U of x. Taking a basis $\{s_i\}$ in T_xM and doing parallel transport along radial geodesics induces a local frame $s = (s_1,\ldots,s_n) : U \to \mathcal{P}^g$ which we call synchronous basis at x. The advantage of this local frame is that covariant derivatives (Levi-Civita) of these vector fields s_i vanish at $x \in M$.

The next proposition collects some useful formulas:

Proposition 2.3 *Let $X,Y,Z,W,U,V \in \mathfrak{X}(M)$. The Bianchi identity and the differential Bianchi identity of the Riemann curvature tensor are*

$$\begin{aligned}0 =& \mathcal{R}(X,Y,Z,W) + \mathcal{R}(Y,Z,X,W) + \mathcal{R}(Z,X,Y,W)\\ 0 =& (\nabla_X\mathcal{R})(Y,Z,W,U) + (\nabla_Y\mathcal{R})(Z,X,W,U) + (\nabla_Z\mathcal{R})(X,Y,W,U)\end{aligned}$$

The action of the curvature tensor on the Schouten and Cotton tensor is given by

$$\begin{aligned}(R(X,Y)P)(U,V) =& -P(R(X,Y)U,V) - P(U,R(X,Y)V)\\ (R(X,Y)C)(U,V,Z) =& -C(R(X,Y)U,V,Z) - C(U,R(X,Y)V,Z)\\ &- C(U,V,R(X,Y)Z).\end{aligned}$$

The co-differentials of the Schouten, Weyl and Bach tensors are given by

$$\begin{aligned}(\delta^{\nabla^{LC}}P) =& -dJ,\\ (\delta W)(X,Y,Z) =& -(n-3)C(Z,Y,X),\\ (\delta^{\nabla^{LC}}B)(X) =& (n-4)g(P,C(\cdot,X,\cdot)),\end{aligned}$$

where we have considered the Schouten and the Bach tensor as elements of $\Omega^1(M,T^*M)$, e.g. $\mathfrak{X}(M) \ni X \mapsto P(X,\cdot)$.

Proof. These equation are well known, and proof thereof can be found in [KN63][Theorem

5.3] concerning the first and second Biachi identity and in [Juh09a][Lemma 4.2.7] concerning the co-differentials of the Schouten and Weyl tensor. The action of the curvature tensor on the Schouten and Cotton tensor is a straightforward calculation. The co-differential of the Bach tensor is not easy to find in the literature, therefore we will prove it here. Taking a normal neighbourhood $U \subset M$ around $x \in M$, a synchronous basis $\{s_i\}$ at $x \in M$ and extend $X \in T_xM$ to a vector field over U by parallel transport along radial geodesics. The one has

$$\begin{aligned}
(\delta^{\nabla^{LC}} B)(X) =& -\sum_j \varepsilon_j (\nabla_{s_j} B)(s_j) \\
=& -\sum_{i,j} \varepsilon_i\varepsilon_j \nabla_{s_j}\nabla_{s_i} C(s_i, s_j, X) - \sum_{i,j,k} \varepsilon_i\varepsilon_j\varepsilon_k \nabla_{s_j} \left(P(s_i, s_k) W(s_i, s_j, X, s_k)\right) \\
=& \frac{1}{2}\sum_{i,j} \varepsilon_i\varepsilon_j R(s_j, s_i) C(s_i, s_j, X) + \frac{1}{2}\sum_{i,j,k} \varepsilon_i\varepsilon_j\varepsilon_k C(s_j, s_i, s_k) W(s_i, s_j, X, s_k) \\
& + \sum_{i,k} \varepsilon_i\varepsilon_k \delta W(s_i, X, s_k) P_{ik} \\
=& \frac{1}{2}\sum_{i,j,k} \varepsilon_i\varepsilon_j\varepsilon_k C(s_i, s_j, s_k)(-\mathcal{R}(s_j, s_i, X, s_k) + W(s_j, s_i, X, s_k) \\
& - (n-3)\sum_{i,k} \varepsilon_i\varepsilon_k C(s_k, X, s_i) P_{ik} \\
=& \sum_{i,k} \left[\varepsilon_i\varepsilon_k C(s_i, X, s_k) P_{ik} - (n-3) C(s_k, X, s_i) P_{ik}\right],
\end{aligned}$$

which proves the claim. □

The tensors we have introduced so far are (local) Riemannian invariants, i.e., their local coefficients are polynomials in $det(g)^{-1}$ and $\partial^\alpha g_{ij}$ for a multi-index α which are independent of the chosen coordinates, and they are invariant under isometries.

Spin geometry: Let (M, g) be a semi Riemannian manifold. The definition of the spinor bundle involves some algebraic and geometric considerations. Let us denote by $\mathcal{C}_{p,q}$ the Clifford algebra of $\mathbb{R}^n$ equipped with the Euclidean scalar product $\langle \cdot, \cdot \rangle_{p,q}$ of signature (p, q), i.e., it can be realized as $\mathcal{C}_{p,q} := T(\mathbb{R}^n)/\{x \otimes x + \langle x, x\rangle_{p,q}\}$, where $T(\mathbb{R}^n)$ denotes the tensor algebra of $\mathbb{R}^n$. This algebra carries a unique (up to equivalence) irreducible representation $\Xi_{p,q} : \mathcal{C}_{p,q} \to End(\mathbb{C}^{2^m})$ in case $p+q = 2m$ is even, whereas in case $p+q = 2m+1$ there are exactly two (up to equivalence) non-equivalent irreducible representations $\Xi^0_{p,q}, \Xi^1_{p,q} : \mathcal{C}_{p,q} \to End(\mathbb{C}^{2^m})$. The spin representation $\kappa_{p,q} : Spin(p, q) \to Gl(\mathbb{C}^{2^m})$ is defined by restriction to the spin group $Spin(p, q) \subset \mathcal{C}^0_{p,q}$ of an irreducible representation of $\mathcal{C}_{p,q}$ on $\mathbb{C}^{2^m}$. In case of $p + q = 2m + 1$ it does not matter which irreducible representation we have chosen, because they become equivalent when restricted to $\mathcal{C}^0_{p,q}$. As a matter of facts in case $p+q = 2m$ even, $\kappa_{p,q}$ decomposes into two non-equivalent irreducible representations on $\mathbb{C}^{2^{m-1}}$, whereas in case $p+q = 2m+1$ odd, $\kappa_{p,q}$ is irreducible on $\mathbb{C}^{2^m}$. These data always exist.

Now we come to the geometric part. In what will follow, we will always assume that the semi Riemannian manifold (M, g) is time- and space oriented, hence the frame bundle $(GL(M), \pi, M, Gl(n, \mathbb{R}))$ reduces to the structure group $SO_0(p, q)$, where the subscript $\cdot_0$ attached to a Lie group denotes its connected component containing the identity. Let us denote by $\lambda : Spin_0(p, q) \to SO_0(p, q)$ the twofold covering of $SO_0(p, q)$. A pair $(\mathcal{Q}^g, f^g)$ defines a spin structure of (M, g) if $(\mathcal{Q}^g, f^g)$ is a $\lambda-$reduction of $\mathcal{P}^g$. We call (M, g) a spin manifold if it carries a spin structure. This structure does not exist in general. However, under a topological assumption, the second Stiefel-Withney class has to vanish, they do exist. For details see [LM89] and [Bau81].

Let (M^n, g) be a spin manifold and denote by $(\mathcal{Q}^g, f^g)$ a spin structure on (M, g). The spin representation $\kappa_{p,q} : Spin_0(p, q) \to Gl(\Delta_{p,q})$ gives rise to the spinor bundle

$$S(M, g) := \mathcal{Q}^g \times_{(Spin_0(p,q), \kappa_{p,q})} \Delta_{p,q}$$

of (M, g). Due to the isomorphism $TM \simeq \mathcal{Q}^g \times_{(Spin_0(p,q), \rho\circ\lambda)} \mathbb{R}^n$, we may define the Clifford multiplication between a vector $X \in TM$ and a spinor $\phi \in S(M, g)$ by the following:

$$X \cdot \phi := \begin{cases} [q, \Xi_{p,q}(x)v], & n = 2m, \\ [q, \Xi^0_{p,q}(x)v], & n = 2m + 1 \end{cases},$$

where $X = [q, x]$ and $\phi = [q, v]$. Note, as a matter of conventions, that we also could take the representation $\Xi^1_{p,q}$ in the odd-dimensional case. Note that for any $p-$form $w \in \Omega^p(M)$ and for any spinor field $\phi \in \Gamma(S(M, g))$ their Clifford multiplication is given, locally, by

$$(w \cdot \phi)_{|U} = \sum_{i_1<\ldots<i_p} \varepsilon_{i_1} \ldots \varepsilon_{i_p} w(s_{i_1}, \ldots, s_{i_p}) s_{i_1} \cdot \ldots \cdot_{i_p} \cdot \phi, \tag{2.2}$$

where $s = \{s_i\} : U \to \mathcal{P}^g$ is a local section. In case of a $2-$form η we have

$$(\eta \cdot \phi)_{|U} = \sum_{i<j} \varepsilon_i \varepsilon_j \eta(s_i, s_j) s_i \cdot s_j \cdot \phi = \frac{1}{2} \sum_{i\neq j} \varepsilon_i \varepsilon_j \eta(s_i, s_j) s_i \cdot s_j \cdot \phi.$$

Since λ is a covering, the Levi-Civita connection form $A^g \in \Omega^1(\mathcal{P}^g, \mathfrak{so}(p, q))$ induces a connection $\tilde{A}^g := \lambda_*^{-1} \circ A^g \circ df^g \in \Omega^1(\mathcal{Q}^g, \mathfrak{spin}(p, q))$, which induces a covariant derivative $\nabla^{S(M,g)}$ on the spinor bundle $S(M, g)$. Locally, it is given by

$$\begin{aligned}(\nabla^{S(M,g)}_X \phi)_{|U} =& [q, dv_q(X) + \rho_*((\tilde{A}^g)^q(X))v] \\ =& X(\phi) + \frac{1}{2} \sum_{i<j} \varepsilon_i \varepsilon_j g(\nabla^{LC}_X s_i, s_j) s_i \cdot s_j \cdot \phi,\end{aligned}$$

where $\phi = [q, v]$ for a local section $q : U \to \mathcal{Q}^g$, $s = (s_1, \ldots, s_n) := f^g(q)$ and a smooth map $v : U \to \Delta_{p,q}$. Furthermore, there is a bundle metric $< \cdot, \cdot >$ on $S(M, g)$, essentially induced by the Hermitian scalar product of $\mathbb{C}^N$ for appropriate N and a little technical issue for pseudo Riemannian structures. In case of signature $(0,n)$ this is positive definite, whereas in case of (p,q), $p,q \neq 0$ this is indefinite. Let us collect some basic properties (compare [LM89], or [Bau81]).

Proposition 2.4 *Let $\phi, \psi \in \Gamma(S(M,g))$ and $X, Y \in \mathfrak{X}(M)$. Then one has*

(1) $\nabla_X^{S(M,g)}(Y \cdot \phi) = \nabla_X^{LC} Y \cdot \phi + Y \cdot \nabla_X^{S(M,g)} \phi$,

(2) $\nabla^{S(M,g)}$ *is metric with respect of* $< \cdot, \cdot >$,

(3) $< X \cdot \phi, \psi > + (-1)^p < \phi, X \cdot \psi > = 0$,

(4) $\mathcal{R}^{S(M,g)}(X,Y)\phi = \frac{1}{2}\mathcal{R}(X,Y) \cdot \phi$ *and*

(5) $\sum_{k=1}^n \varepsilon_k s_k \cdot \mathcal{R}^{S(M,g)}(s_k, Y)\phi = \frac{1}{2} Ric(Y)^\natural \cdot \phi$,

where we have considered the Riemann curvature tensor as element in $\Omega^2(M, \Lambda^2 M)$ and the Ricci tensor as element in $\Omega^1(M, T^*M)$

The Dirac operator $D\!\!\!/$ associated to a spin manifold (M,g) is given by the composition:

$$\begin{aligned} D\!\!\!/ : \Gamma(S(M,g)) \xrightarrow{\nabla^{S(M,g)}} \Gamma(T^*M \otimes S(M,g)) &\xrightarrow{\cdot^\natural} \Gamma(TM \otimes S(M,g)) \xrightarrow{\mu} \Gamma(S(M,g)) \\ \phi &\mapsto D\!\!\!/ \phi := \mu((\nabla^{S(M,g)}\phi)^\natural). \end{aligned}$$

Note that the Dirac operator depends on the semi Riemannian metric, since the spin connection and Clifford multiplication are involved in its definition. For any section $\phi \in \Gamma(S(M,g))$ and local section $s : U \to \mathcal{P}^g$ the Dirac operator reads, locally,

$$(D\!\!\!/ \phi)_{|U} = \sum_i \varepsilon_i s_i \cdot \nabla_{s_i}^{S(M,g)} \phi.$$

Furthermore, the Dirac operator fulfills the product rule

$$[D\!\!\!/, f]\phi = D\!\!\!/(f\phi) - f D\!\!\!/ \phi = \mathrm{grad}^g(f) \cdot \phi$$

for all $f \in \mathcal{C}^\infty(M)$ and $\phi \in \Gamma(S(M,g))$. As a consequence of formulas from Proposition 2.4, we may derive

$$\begin{aligned} [D\!\!\!/, \nabla_X^{S(M,g)}]\phi =& \frac{1}{2} Ric(X)^\natural \cdot \phi \\ D\!\!\!/^2 \phi =& -\Delta_g^{S(M,g)}\phi + \frac{\tau}{4}\phi \end{aligned} \tag{2.3}$$

for all $X \in \mathfrak{X}(M)$, $\phi \in \Gamma(S(M,g))$ and $\Delta_g^{S(M,g)} := \mathrm{tr}_g(\nabla^{T^*M \otimes S(M,g)} \circ \nabla^{S(M,g)})$. The latter formula is the Weitzenböck formulas for the square of the Dirac operator, compare [LM89][Theorem 8.8]. Note the sign convention we used for $\Delta_g^{S(M,g)}$.

In order to present a compact notation of the first conformal powers of the Dirac operator, it is useful to fix some notations: For symmetric $(0,2)-$tensor fields T, T_1, T_2,

$\phi \in \Gamma\left(S(M,g)\right)$ and $\eta \in \Omega^1\left(M, S(M,g)\right)$, we define

$$\begin{aligned}(T,\eta) :=&\mu\left(\mathrm{tr}_g(T(\cdot)^\natural \otimes \eta)\right) \in \Gamma\left(S(M,g)\right) \\ \overset{\text{loc.}}{=}& \sum_i \varepsilon_i T(s_i)^\natural \cdot \eta(s_i),\end{aligned} \tag{2.4}$$

where $\overset{loc}{=}$ indicates the local representation with respect to a local section $s = (s_1, \ldots, s_n) : U \subset M \to \mathcal{P}^g$. From T_1 and T_2 one can define a symmetric $(0,2)-$tensor field $T_1 \cdot T_2$ by

$$(T_1 \cdot T_2)(X,Y) := \frac{1}{2}\left(T_1(T_2(Y)^\natural, X) + T_1(T_2(X)^\natural, Y)\right), \tag{2.5}$$

where $X, Y \in \mathfrak{X}(M)$. Furthermore, we define a $1-$form $T \cdot \phi$ with values in the spinor bundle by

$$(T \cdot \phi)(X) := T(X)^\natural \cdot \phi. \tag{2.6}$$

Due to aesthetical reasons let us introduce the following

$$\begin{aligned}(\nabla^{S(M,g)}, T \cdot \phi) :=& -\delta^{\nabla^{S(M,g)}}(T \cdot \phi). \\ \overset{\text{loc.}}{=}& \sum_i \varepsilon_i \nabla^{S(M,g)}_{s_i}(T \cdot \phi)(s_i)\end{aligned} \tag{2.7}$$

The brackets defined above satisfy

$$(\nabla^{S(M,g)}, T \cdot \phi) = -(\delta^{\nabla^{LC}} T)^\natural \cdot \phi + (T, \nabla^{S(M,g)}\phi). \tag{2.8}$$

Note the Dirac operator written in this notation is

$$D\!\!\!/ = \frac{1}{2}\left[(g, \nabla^{S(M,g)}) + (\nabla^{S(M,g)}, g\cdot)\right].$$

On spin manifolds (M,g) without boundary, it is well known that the Dirac operator is formally self-adjoint or formally anti self-adjoint with respect to the L^2-scalar product $< \phi, \psi >_{L^2} := \int_M < \phi(x), \psi(x) > dM_g$ on the spinor bundle, depending on the sign in $< X \cdot \phi, \psi > +(-1)^p < \phi, X \cdot \psi >= 0$, where $X \in \mathfrak{X}(M)$ and $\phi, \psi \in \Gamma_c\left(S(M,g)\right)$. The bracket notation reflects this fact in a trivial way, in view of the following proposition:

Proposition 2.5 *Let (M,g) be a spin manifold without boundary and T a symmetric $(0,2)$-tensor. The operator*

$$(T, \nabla^{S(M,g)}) + (\nabla^{S(M,g)}, T\cdot) : \Gamma\left(S(M,g)\right) \to \Gamma\left(S(M,g)\right)$$

is formally self-adjoint (anti self-adjoint) with respect to the L^2-scalar product, i.e.,

$$< (T, \nabla^{S(M,g)})\phi + (\nabla^{S(M,g)}, T\cdot)\phi, \psi >_{L^2}$$

$$-(-1)^p < \phi, (T, \nabla^{S(M,g)})\psi + (\nabla^{S(M,g)}, T\cdot)\psi >_{L^2} = 0,$$

where $\phi, \psi \in \Gamma_c(S(M,g))$.

Proof. For $\phi, \psi \in \Gamma_c(S(M,g))$ we define a $1-$form with complex values by

$$w(X) := < T(X)^\natural \cdot \phi, \psi > .$$

Consider the vector field Y_w dual to w and compute

$$\begin{aligned} div(Y_w) =& \sum_{i=1}^{n} \epsilon_i [< T(s_i)^\natural \cdot \nabla^{S(M,g)}_{s_i}\phi, \psi > -(-1)^p < \phi, T(s_i)^\natural \cdot \nabla^{S(M,g)}_{s_i}\psi >] \\ &+ (-1)^p < \phi, (\delta^{\nabla^{LC}} T)^\natural \cdot \psi > . \end{aligned} \tag{2.9}$$

From Stokes' Theorem we get that $\int_M div(Y_w) dM = 0$. Together with equation (2.8) and equation (2.9) this yields

$$\begin{aligned} &<(T, \nabla^{S(M,g)})\phi + (\nabla^{S(M,g)}, T\cdot)\phi, \psi >_{L_2} \\ =& \int_M < (T, \nabla^{S(M,g)})\phi + (\nabla^{S(M,g)}, T\cdot)\phi, \psi > dM_g \\ =& \int_M < 2(T, \nabla^{S(M,g)})\phi - (\delta^{\nabla^{LC}} T)^\natural \cdot \phi, \psi > dM_g \\ =& \int_M (-1)^p < \phi, 2(T, \nabla^{S(M,g)})\psi - 2(-1)^p (\delta^{\nabla^{LC}} T)^\natural \cdot \psi > dM_g \\ &- \int_M < (\delta^{\nabla^{LC}} T)^\natural \cdot \phi, \psi > dM_g \\ =&(-1)^p \int_M < \phi, 2(T, \nabla^{S(M,g)})\psi - (\delta^{\nabla^{LC}} T)^\natural \cdot \psi > dM_g \\ =&(-1)^p \int_M < \phi, (T, \nabla^{S(M,g)})\psi + (\nabla^{S(M,g)}, T\cdot)\psi > dM_g. \end{aligned}$$

This proves the lemma. □

This lemma will ensure that on a spin manifold (M, g) without boundary the lower conformal powers of the Dirac operator on compactly supported sections become formally (anti-) self-adjoint with respect to the L^2-scalar product, due to explicit formulas.

Later on, complicated computations will require the following technical lemma.

Lemma 2.6 *Let A be a $(0,3)-$tensor field such that $A(X, Y, Z) = -A(Y, X, Z)$ and $A(X, Y, Z) + A(Y, Z, X) + A(Z, X, Y) = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$. Then one has*

$$\sum_{i \neq j,k} \varepsilon_i \varepsilon_j \varepsilon_k A(s_i, s_j, s_k) s_i \cdot s_j \cdot s_k = -2 \sum_{i \neq k} \varepsilon_i \varepsilon_k A(s_i, s_k, s_k) s_i$$

and

$$2\sum_{k,l}\varepsilon_k\varepsilon_l A(s_k,X,s_l)s_k\cdot s_l=\sum_{k,l}\varepsilon_k\varepsilon_l A(s_k,s_l,X)s_k\cdot s_l+2\sum_k\varepsilon_k A(X,s_k,s_k),$$

where $\{s_i\}$ *is a* $g-$*orthonormal basis.*

The first fact was already used to derive some results in Proposition 2.4, but because of its importance we will present this fact individually.

Proof. We decompose the left hand side of the first equation as follows:

$$\sum_{i\neq j,k=j}\varepsilon_i\varepsilon_j\varepsilon_k A(s_i,s_j,s_k)s_i\cdot s_j\cdot s_k+\sum_{i\neq j,k=i}\varepsilon_i\varepsilon_j\varepsilon_k A(s_i,s_j,s_k)s_i\cdot s_j\cdot s_k$$
$$+\sum_{i\neq j,k\neq i,k\neq j}\varepsilon_i\varepsilon_j\varepsilon_k A(s_i,s_j,s_k)s_i\cdot s_j\cdot s_k.$$

The last summand vanishes due to $s_i\cdot s_j=-s_j\cdot s_i$ for $i\neq j$,

$$\begin{aligned}\sum_{i\neq j,k\neq i,k\neq j}&\varepsilon_i\varepsilon_j\varepsilon_k A(s_i,s_j,s_k)s_i\cdot s_j\cdot s_k=\\&=-\sum_{i\neq j,k\neq i,k\neq j}\varepsilon_i\varepsilon_j\varepsilon_k\big(A(s_j,s_k,s_i)+A(s_k,s_i,s_j)\big)s_i\cdot s_j\cdot s_k\\&=-2\sum_{i\neq j,k\neq i,k\neq j}\varepsilon_i\varepsilon_j\varepsilon_k A(s_i,s_j,s_k)s_i\cdot s_j\cdot s_k.\end{aligned}$$

Thus the first equation follows from $s_i\cdot s_i=-\varepsilon_i$.

The second claim is proved as follows:

$$\begin{aligned}\sum_{k,l}\varepsilon_k\varepsilon_l A(s_k,X,s_l)s_k\cdot s_l&=-\sum_{k,l}\varepsilon_k\varepsilon_l\big(A(X,s_l,s_k)+A(s_l,s_k,X)\big)s_k\cdot s_l\\&=\sum_{k,l}\varepsilon_k\varepsilon_l\big(-A(s_k,X,s_l)+A(s_k,s_l,X)\big)s_k\cdot s_l+2\sum_k\varepsilon_k A(X,s_k,s_k).\end{aligned}$$

The trace shows up because we have started with a sum over k,l. □

Candidates satisfying the properties of the last lemma are given by the Riemannian curvature, Weyl, and Cotton tensor.

Cartan geometry: Felix Klein's idea to model a geometry as Klein geometry

$$(G,\pi,G/H,H;w_G),$$

where G is a Lie group, $H\subset G$ is a closed subgroup and $w_G:TG\to\mathfrak{g}$ is the Maurer-Cartan form of G, given by $w_G(X_g):=(dL_{g^{-1}})_g(X_g)$, where $L_{g^{-1}}:G\to G$ denotes left translation inside G, will be generalized to curved manifolds through the concept of Cartan geometries. Let M be a manifold, G a Lie group and $H\subset G$ a closed subgroup. A Cartan geometry of type (G,H) on a manifold M is a pair $(\mathcal{G},w)$ which consists of an $H-$principal bundle $(\mathcal{G},\pi,M,H)$ and a Cartan connection $w\in\Omega^1(\mathcal{G},\mathfrak{g})$, i.e., $w_p(\tilde{X}(p))=X$ for all $X\in\mathfrak{g}$, $(R_h^*w)_p(X)=Ad(h^{-1})w_{h\cdot p}(X)$ and $w_p:T_p\mathcal{G}\to\mathfrak{g}$ is an isomorphism, where $p\in\mathcal{G},h\in H$ and $X\in\mathfrak{X}(\mathcal{G})$. The last property of a Cartan connection gives rise for the notion of $w-$constant vector fields $w^{-1}(X)\in\mathfrak{X}(\mathcal{G})$ associated to $X\in\mathfrak{g}$ given by $w^{-1}(X)(p):=w_p^{-1}(X)$. The curvature $K\in\Omega^2(\mathcal{G},\mathfrak{g})$ of a Cartan connection is defined by $K(X,Y):=dw(X,Y)+[w(X),w(Y)]_{\mathfrak{g}}$ for all $X,Y\in T\mathcal{G}$. Because of the isomorphism property of a Cartan connection the curvature K can be equivalently described by its curvature function $k\in\mathcal{C}^\infty(\mathcal{G},\Lambda^2\mathfrak{g}^*\otimes\mathfrak{g})$ defined by $k(p)(X,Y):=K(w^{-1}(X)(p),w^{-1}(Y)(p))$ for $X,Y\in\mathfrak{g}$. Let us state two examples of Cartan geometries:

Example 2.7 Let M be a manifold. The frame bundle $(GL(M),\pi,M,Gl(n,\mathbb{R}))$ carries a canonical form $\theta\in\Omega^1(GL(M),\mathbb{R}^n)$ defined by $\theta(e)(\xi):=[e]^{-1}(d\pi_e(\xi))$, where $e=(e_1,\dots,e_n)\in GL(M)$, $[e]:\mathbb{R}^n\to T_{\pi(e)}M$ is the isomorphism $[e]v:=\sum_i v^ie_i$ and $\xi\in T_eGL(M)$, satisfying $R_g^*\theta=\rho(g^{-1})\theta$ and $\theta(\xi)=0$ for $g\in Gl(n,\mathbb{R})$ and $\xi\in Tv_eGL(M)$. A covariant derivative ∇ on the tangent bundle gives rise to a connection form $A^\nabla\in\Omega^1(GL(M),\mathfrak{gl}(n,\mathbb{R}))$ induced by ∇. These data can be encoded in a special Cartan geometry. Namely, consider the Lie-group

$$A(n,\mathbb{R}):=\left\{\left.\begin{pmatrix}1&0\\b&A\end{pmatrix}\right|A\in Gl(n,\mathbb{R}),b\in\mathbb{R}^n\right\}\subset Gl(n+1,\mathbb{R})$$

with Lie algebra

$$\mathfrak{a}(n,\mathbb{R}):=\left\{\left.\begin{pmatrix}0&0\\X&B\end{pmatrix}\right|B\in\mathfrak{gl}(n,\mathbb{R}),X\in\mathbb{R}^n\right\}\subset\mathfrak{gl}(n+1,\mathbb{R})=Mat(n+1,\mathbb{R}).$$

The latter is isomorphic to $\mathbb{R}^n\oplus\mathfrak{gl}(n,\mathbb{R})$ as a vector space. From the given data we may define a Cartan connection $w:=\theta+A^\nabla$ on the frame bundle $Gl(M)$ of type $(A(n,\mathbb{R}),Gl(n,\mathbb{R}))$ and thus a Cartan geometry $(GL(M),w)$ on M. Conversely, any Cartan geometry $(GL(M),w)$ of type $(A(n,\mathbb{R}),Gl(n,\mathbb{R}))$ on a manifold M induces a connection form $A\in\Omega^1(GL(M),\mathfrak{gl}(n,\mathbb{R}))$, hence a covariant derivative on the tangent bundle.

Example 2.8 Let (M,g) be a semi Riemannian manifold. Any $g-$metric covariant derivative ∇ on the tangent bundle induces a connection form $A^g\in\Omega(\mathcal{P}^g,\mathfrak{o}(n,\mathbb{R}))$ on the orthonormal frame bundle $\mathcal{P}^g$. Thus the procedure given in the last example shows that the structure (M,g,∇) describes a Cartan geometry $(\mathcal{P}^g,w=\theta_{|\mathcal{P}^g}+A^g)$ of type $(Euc(p,q,\mathbb{R}),O(p,q))$, where

$$Euc(p,q,\mathbb{R}):=\left\{\left.\begin{pmatrix}1&0\\b&A\end{pmatrix}\right|A\in O(p,q),b\in\mathbb{R}^n\right\}\subset Gl(n+1,\mathbb{R})$$

is the group of euclidean motions. Conversely, any Cartan geometry $(\mathcal{P}^g, w)$ of type $(Euc(p,q,\mathbb{R}), O(p,q))$ gives rise to a connection form A in $\mathcal{P}^g$.

Now, let $(\mathcal{G}, w)$ be a Cartan geometry of type (G, H) on M, and consider a representation $\rho : G \to Gl(V)$ of G on some finite-dimensional vector space V. Restricting the representation to the subgroup H leads to the associated vector bundle

$$\mathcal{T} := \mathcal{G} \times_{(H,\rho)} V,$$

which is called tractor bundle of M. Though a Cartan connection w does not yield a covariant derivative on general to $\mathcal{G}$ associated vector bundles, we can define a covariant derivative on the tractor bundle induced by w.

Lemma 2.9 *Let $(\mathcal{G}, w)$ be a Cartan geometry of type (G, H) on M. There exists a covariant derivative $\nabla^{\mathcal{T}}$ on the tractor bundle which is canonically induced by w. Locally, it is given by*

$$(\nabla^{\mathcal{T}}_X t)_{|U} = \Big[u, X(v) + \rho_*\Big(w\big(du(X)\big)\Big)v\Big],$$

where $X \in \mathfrak{X}(M)$, $t \in \Gamma(\mathcal{T})$, and $t = [u, v]$ for a local section $u : U \to \mathcal{G}$ and a smooth map $v : U \to V$.

Proof. Let us denote by $\pi_{\mathcal{G}}$ and π_G the projection mappings from $\mathcal{G} \times G$ to $\mathcal{G}$ and G. The Cartan connection induces a 1−form with values in the Lie algebra of G by

$$\tilde{w}_{(u,g)} := Ad(g^{-1})(\pi^*_{\mathcal{G}} w)_{(u,g)} + (\pi^*_G w_G)_{(u,g)},$$

where w_G is the Maurer-Cartan form of G and $(u, g) \in \mathcal{G} \times G$. One computes that $\tilde{w}$ is invariant with respect to the H−action

$$\begin{aligned} \alpha : H \times (\mathcal{G} \times G) &\to \mathcal{G} \times G \\ (h, (u, g)) &\mapsto \alpha_h(u, g) := (u \cdot h, h^{-1} \cdot g), \end{aligned}$$

i.e., $(\alpha^*_h \tilde{w})_{(u,g)} = \tilde{w}_{(u,g)}$ for all $h \in H$ and $(u, g) \in \mathcal{G} \times G$. Hence, we can project $\tilde{w}$ to the extension $\bar{\mathcal{G}} := \mathcal{G} \times_H G$ of the H−principal bundle $\mathcal{G}$, i.e., $\bar{w} \in \Omega^1(\mathcal{G} \times_H G, \mathfrak{g})$ is given by $\bar{w}_{[u,g]} := \tilde{w}_{(u,g)}$ for all $[u, g] \in \mathcal{G} \times_H G$. Since,

$$\begin{aligned} (R^*_a \bar{w})_{[u,g]} &= Ad(a^{-1})\bar{w}_{[u,g]}, \\ \bar{w}_{[u,g]}(\tilde{X}) &= X, \end{aligned}$$

where $[u, g] \in \mathcal{G} \times_H G$, $a \in G$ and $\tilde{X} \in \mathfrak{X}(\bar{\mathcal{G}})$ is the fundamental vector field of $X \in \mathfrak{g}$, it follows that $\bar{w} \in \Omega^1(\bar{\mathcal{G}}, \mathfrak{g})$ is a connection form on $(\bar{\mathcal{G}}, \pi, M, G)$. Considering the inclusion map $\iota : \mathcal{G} \to \bar{\mathcal{G}}$, given by $\iota(u) := [u, e]$, yields $\iota^* \bar{w} = w$. From the isomorphism

$$\mathcal{T} = \mathcal{G} \times_{(H,\rho)} V \simeq \bar{\mathcal{G}} \times_{(G,\rho)} V$$

we may define $\nabla^w := \nabla^{\bar{w}}$ on $\mathcal{T}$, where $\nabla^{\bar{w}}$ is the covariant derivative induced from the connection form $\bar{w} \in \Omega^1(\bar{\mathcal{G}}, \mathfrak{g})$. Locally, ∇^w is given by

$$(\nabla^w_X t)_{|U} = \Big[u, X(v) + \rho_* \Big(w\big(du(X)\big)\Big) v\Big] \, ,$$

where $X \in \mathfrak{X}(M)$, $t \in \Gamma(\mathcal{T})$, and $t = [u, v]$ for a local section $u : U \to \mathcal{G}$ and a smooth function $v : U \to V$. $\square$

As ususal, any $H-$invariant scalar product on V, in particular, any $G-$invariant scalar product on V, induces a bundle metric on the tractor bundle.

Thus we have finished the introduction of standard objects we will work with in this thesis.

3 Conformal geometry

In semi Riemannian geometry one studies manifolds equipped with one arbitrary semi Riemannian metric. Considering an equivalence class of conformally equivalent metrics leads us to the realm where conformal geometry lives.

Let M be a smooth manifold equipped with a semi Riemannian metric g. A metric h on M is said to be conformally equivalent to g if there exists a function $\sigma \in \mathcal{C}^{\infty}(M)$ such that $h = e^{2\sigma} g$. Clearly, this defines an equivalence relation.

Definition 3.1 *A conformal structure $c := [g]$ on a manifold M is an equivalence class of conformally equivalent metrics on M.*

We will always assume that our manifolds in question have dimension ≥ 3.

Remark 3.2 Let (M, g) be a simply connected $2-$dimensional oriented Riemannian manifold. The uniformization Theorem asserts that (M, g) is conformally equivalent to one of the following three spaces; the complex plane, the open unit disc or the Riemannian sphere.

The first two sections will deal with the relation of semi Riemannian and spin calculus with respect to conformally equivalent metrics. Realising that describing conformal geometry hardly makes sense with the tools of semi Riemannian geometry, the goal of the next sections is to introduce an invariant calculus, which only depends on the conformal structure and not on any chosen representative of the conformal structure.

3.1 Conformal behavior of semi Riemannian geometry

Let (M, c) be a conformal manifold. Choosing two metrics $g, \hat{g} \in c$ gives us two Levi-Civita connections ∇^{LC} and $\hat{\nabla}^{LC}$. The $\hat{}$ indicates that the quantity in question is induced by $\hat{g}$.

Lemma 3.3 *Let $g, \hat{g} = e^{2\sigma} g \in c$. Their Levi-Civita connections are related by*

$$\hat{\nabla}^{LC}_X Y = \nabla^{LC}_X Y + X(\sigma)Y + Y(\sigma)X - g(X, Y)\,\mathrm{grad}^g(\sigma),$$

where $X, Y \in \mathfrak{X}(M)$.

Proof. To prove this property, we will express the Christoffel symbols with respect to $\hat{g}$ in terms of those with respect to g. Let $(x_1, \ldots x_n)$ be a coordinate chart on $U \subset M$

and denote by $\{\partial_i\}_{i=1}^n$ the induced basis in the tangent bundle. First one has

$$\Gamma_{ij}^k = \frac{1}{2}\sum_l g^{kl}(\partial_i(g_{jl}) + \partial_j(g_{il}) - \partial_l(g_{ij})),$$

hence

$$\Gamma_{ij}^{\hat{g},k} = \Gamma_{ij}^{g,k} + \partial_i(\sigma)\delta_{jk} + \partial_j(\sigma)\delta_{ik} - \sum_l g^{kl} g_{ij}\partial_j(\sigma),$$

where δ_{ik} denotes the Kronecker delta. Finally,

$$\hat{\nabla}_X^{LC} Y = \sum_{i,j} X^i \left(\sum_k Y^j \Gamma_{ij}^{\hat{g},k}\partial_k + \partial_i(Y^j)\partial_j\right),$$

completes the proof. $\square$

A consequence of Lemma 3.3 is a collection of transformation formulas for the tensors we have defined before. In order to shorten equations, we define a $(0,2)-$tensor field on M by

$$\Xi^{g,\sigma}(X,Y) := \mathrm{Hess}^g(\sigma)(X,Y) - X(\sigma)Y(\sigma) + \frac{1}{2}|\mathrm{grad}^g(\sigma)|_g^2 g(X,Y),$$

where $|\mathrm{grad}^g(\sigma)|_g^2 := g(\,\mathrm{grad}^g(\sigma), \mathrm{grad}^g(\sigma))$, $Hess^g(\sigma)(X,Y) := g(\nabla_X^{LC}\,\mathrm{grad}^g(\sigma), Y)$ and $X, Y \in \mathfrak{X}(M)$.

Lemma 3.4 *Let $\hat{g} = e^{2\sigma}g$ and $X, Y, Z, W \in \mathfrak{X}(M)$. Then one has*

$$\begin{aligned}
\mathcal{R}^{\hat{g}}(X,Y,Z,W) =& e^{2\sigma}\left[\mathcal{R}^g(X,Y,Z,W) + (\Xi^{g,\sigma} \owedge g)(X,Y,Z,W)\right],\\
e^{2\sigma} J^{\hat{g}} =& J^g - \Delta_g(\sigma) - \frac{n-2}{2}|\mathrm{grad}^g(\sigma)|_g^2, \quad \Big(\Delta_g(\sigma) = \mathrm{tr}_g\left(\mathrm{Hess}^g(\sigma)\right)\Big),\\
P^{\hat{g}}(X,Y) =& P^g(X,Y) - \Xi^{g,\sigma}(X,Y),\\
W^{\hat{g}}(X,Y) =& W^g(X,Y), \text{ considered as endomorphism of one-forms},\\
C^{\hat{g}}(X,Y) =& C^g(X,Y) - W^g(X,Y)(d\sigma), \text{ and finally}\\
e^{2\sigma} B^{\hat{g}}(X,Y) =& B^g(X,Y) - (n-4)\big(C^g(X, \mathrm{grad}^g(\sigma), Y) + C^g(Y, \mathrm{grad}^g(\sigma), X)\big)\\
&+ (n-4) W^g(\,\mathrm{grad}^g(\sigma), X, Y, \mathrm{grad}^g(\sigma)).
\end{aligned}$$

Proof. The transformation of the Riemann curvature tensor is a calculation using the conformal transformation of the Levi-Civita connection, compare [Juh09a][Chapter 2.5]. The law for the normalized scalar curvature J is given by the double trace of the

Riemann curvature tensor:

$$\begin{aligned} J^{\hat{g}} =& \frac{1}{2(n-1)} \sum_{i,j} \varepsilon_i \varepsilon_j \mathcal{R}^{\hat{g}}(\hat{s}_i, \hat{s}_j, \hat{s}_j, \hat{s}_i) \\ =& e^{-2\sigma}\left(J^g - \mathrm{tr}_g(\Xi^{g,\sigma})\right) = e^{-2\sigma}(J^g - \Delta_g \sigma - \frac{n-2}{2} |\mathrm{grad}^g(\sigma)|_g^2). \end{aligned}$$

In case of the Schouten tensor we have

$$\begin{aligned} (n-2)P^{\hat{g}}(X,Y) =& Ric^{\hat{g}} - J^{\hat{g}}\hat{g}(X,Y) \\ =& Ric^g(X,Y) + (2-n)\Xi^{g,\sigma}(X,Y) - \mathrm{tr}_g(\Xi^{g,\sigma})g(X,Y) \\ & - J^g g(X,Y) + \mathrm{tr}_g(\Xi^{g,\sigma})g(X,Y). \end{aligned}$$

The Weyl tensor $W(X,Y)$ considered as endomorphism on $1-$forms, i.e., acting by $(W(X,Y)\eta)(Z) = W(X,Y,\eta^\flat,Z)$, transforms by

$$\begin{aligned} W^{\hat{g}}(X,Y) =& \mathcal{R}^{\hat{g}}(X,Y) + (P^{\hat{g}} \owedge \hat{g})(X,Y) \\ =& \mathcal{R}^g(X,Y) + (\Xi^{g,\sigma} \owedge g)(X,Y) + (P^g \owedge g)(X,Y) - (\Xi^{g,\sigma} \owedge g)(X,Y) \\ =& W^g(X,Y). \end{aligned}$$

The remaining proofs of the conformal transformation laws for the Cotton and Bach tensor are quite computational. We do not show them here, since later on we will derive their transformation laws from the tractor machinery, see Remark 5.11 and Remark 5.35. □

These transformation laws show that there exist Riemannian invariants, which are not invariant under a conformal change of the metric. A (local) Riemannian invariant $C(g)$ is called a (local) conformal invariant of conformal weight $k \in \mathbb{Z}$ if for any conformally related metric $\hat{g} = e^{2\sigma} g$ one has

$$C(\hat{g}) = e^{k\sigma} C(g).$$

The metric itself is a conformal invariant with conformal weight 2. Also, the Weyl tensor is a conformal invariant. Its conformal weight depends on the way of viewing the Weyl tensor: Considered as $(1,3)-$tensor, it has conformal weight 0, whereas as $(0,4)-$tensor, it has conformal weight 2, because of the passing from $(1,3)$-tensor fields to $(0,4)-$tensor fields requires the corresponding metric. Furthermore, in dimension three the Cotton tensor is a conformal invariant with conformal weight 0, since the Weyl tensor vanishes in dimensions less than four. Finally, in dimension four, the Bach tensor is conformally invariant with conformal weight -2.

Remark 3.5 Let (M,g) be an oriented semi Riemannian manifold and consider the conformal class $c = [g]$. Since we cannot change the signature of a metric by a factor $e^{2\sigma}$, we may speak about a signature of a conformal structure. Since orientability is independend of g, it makes sence to speak about oriented conformal structures. Furthermore, causality is also preserved by conformal equivalence. Hence, a time (space) orientation for one metric induces a time (space) orientation for any other conformally equivalent metric. Thus one can speak about a time (space) oriented conformal structures (M, c).

3.2 Conformal behavior of spin geometry

The definition of the spinor bundle of a spin manifold (M, g) requires the metric g. We will give a way to identify quantities living in spinor bundles of two conformally equivalent metrics. We follow [Bau81, LM89].

Let (M, g) be a semi Riemannian spin manifold and let $\hat{g} := e^{2\sigma} g$ be a conformally equivalent metric. The map $L_\sigma : TM \to TM$ defined by $L_\sigma(X) := e^{-\sigma} X$ leads to a principal $SO_0(p, q)-$bundle isomorphism

$$\begin{aligned} \Lambda_\sigma : \mathcal{P}^g &\to \mathcal{P}^{\hat{g}} \\ s = (s_1, \dots, s_n) &\mapsto \Lambda_\sigma(s) := (e^{-\sigma} s_1, \dots, e^{-\sigma} s_n). \end{aligned}$$

A spin structure $(\mathcal{Q}^g, f^g)$ for (M, g) induces through Λ_σ a spin structure $(\mathcal{Q}^{\hat{g}}, f^{\hat{g}})$ for $(M, \hat{g})$ and an isomorphism of $Spin_0(p, q)-$principal bundles $\Phi_\sigma : \mathcal{Q}^g \to \mathcal{Q}^{\hat{g}}$. Since Φ_σ is $Spin_0(p, q)-$equivariant, we obtain a vector bundle isomorphism

$$\begin{aligned} F_\sigma : S(M, g) &\to S(M, \hat{g}) \\ [q, v] &\mapsto F_\sigma([q, v]) := [\Phi_\sigma(q), v]. \end{aligned} \tag{3.1}$$

Thus we have identified spinors of two different spinor bundles with respect to conformally equivalent metrics. For practical reasons, let us define

$$\hat{X} := L_\sigma(X), \quad \hat{\psi} := F_\sigma(\psi)$$

for $X \in TM$ and $\psi \in S(M, g)$. Then:

$$F_\sigma(X \cdot \psi) = \widehat{X \cdot \psi} = \hat{X} \hat{\cdot} \hat{\psi} = L_\sigma(X) \hat{\cdot} F_\sigma(\psi),$$

where $\hat{\cdot}$ is the Clifford multiplication on $S(M, \hat{g})$. Therefore, Clifford multiplication is also well-behaved under conformal change. Recall the local representation of the covariant derivative $\nabla^{S(M,g)}$, equation (2.1), in order to derive the following formula:

$$\nabla_X^{S(M,\hat{g})}(F_\sigma \psi) = F_\sigma(\nabla_X^{S(M,g)} \psi) - \frac{1}{2} F_\sigma([X \cdot \mathrm{grad}^g(\sigma) + X(\sigma)] \cdot \psi). \tag{3.2}$$

This formula shows how the Dirac operators with respect to two conformal equivalent metrics are related, namely by

$$\hat{D\!\!\!/}(e^{\frac{1-n}{2}\sigma} \hat{\psi}) = e^{-\frac{n+1}{2}\sigma} \widehat{D\!\!\!/(\psi)}. \tag{3.3}$$

This is a remarkable identity for Dirac operators associated to conformally equivalent metrics. Later we will give a definition for conformally covariant differential operators of certain bi-degrees where the Dirac operator fits in.

Thus we have derived a suitable calculus on spinor bundles arising from two conformally equivalent metrics.

3.3 The Möbius sphere

This section describes a method how the conformal class $[g_c]$ of the round metric g_c on the standard sphere S^n arises in a different way. For details see [Sch97, Sha97, ČS09].

Consider the light cone C of the pseudo Riemannian manifold $(\mathbb{R}^{n+2}, \langle\cdot,\cdot\rangle_{p+1,q+1})$ which we will denote by $\mathbb{R}^{p+1,q+1}$. The projective light cone $\mathbb{P}C := \{\mathbb{R}\cdot x \mid x \in C\}$ defines a smooth manifold in $\mathbb{R}^{n+2}$ which is compact and carries the canonical projection $\pi : C \to \mathbb{P}C$. A section $\mu : \mathbb{P}C \to C^+$ with respect to π, where C^+ is the positive light cone, defines a metric on $\mathbb{P}C$ of signature (p,q) by $h_\mu := \mu^*\langle\cdot,\cdot\rangle_{p+1,q+1}$. The point is that for another section $\hat{\mu} : \mathbb{P}C \to C^+$ the induced metric $h_{\hat{\mu}}$ is conformal equivalent to h_μ. Therefore, the projective light cone carries a conformal structure in a natural way.

Definition 3.6 *The conformal manifold $(Q^{p,q} := \mathbb{P}C, c := [h_\mu])$ is called the Möbius sphere.*

Calling this space a sphere comes from the fact that in case of Euclidean signature $p = n$ and $q = 0$, the Möbius sphere $(Q^{n,0}, c)$ is diffeomorphic to the round sphere $(S^n, [g_c])$, i.e., $\Phi : S^n \ni x \mapsto \mathbb{R}(1,x) \in \mathbb{P}C$ is a diffeomorphism with inverse given by $\mathbb{P}C \ni \mathbb{R}x \mapsto (\frac{x_1}{x_0}, \ldots, \frac{x_{n+1}}{x_0}) \in S^n$ and the conformal structure $[h_\mu]$ pulls back to the conformal structure of the round metric g_c on S^n, i.e., $\Phi^* h_\mu \in [g_c]$. In case of pseudo Riemannian signature $p \neq 0$ we define a mapping by $\Phi : S^p \times S^q \to \mathbb{P}C$ by $\Phi(z,w) := \mathbb{R}(z,w) \in \mathbb{P}C$ where $z \in S^p$ and $w \in S^q$. This is a twofold covering. Thus the Möbius sphere is diffeomorphic to $(S^p \times S^q)/\{\pm I\}$. The pull back $\Phi^* h_\mu$ lies in the conformal structure induced by the product metric on $S^p \times S^q$.

Proposition 3.7 *There exists an embedding $\iota : \mathbb{R}^{p,q} \to Q^{p,q}$ such that $\iota(\mathbb{R}^{p,q})$ is open and dense in $Q^{p,q}$ and $\langle\cdot,\cdot\rangle_{p,q} \in \iota^* c$, i.e., the Möbius sphere is a conformal compactification of $\mathbb{R}^{p,q}$.*

Proof. Let $\{e_i\}$ be an orthonormal basis of $\mathbb{R}^{p+1,q+1}$ and define a new basis by $f_0 := \frac{1}{\sqrt{2}}(e_{n+1} - e_0)$, $f_i = e_i$ for $i = 1, \ldots, n$ and $f_{n+1} := \frac{1}{\sqrt{2}}(e_{n+1} + e_0)$. The inclusion map is thus given by $\iota(x) := \mathbb{R}(-2 < x > f_0 + 2x + f_{n+1}) \in Q^{p,q}$ for all $x \in \mathbb{R}^{p,q}$ where $< x > := \langle x, x\rangle_{p,q}$. Since f_0 and f_{n+1} are linearly independent, we conclude that $\iota : \mathbb{R}^{p,q} \to Q^{p,q}$ and $d\iota_x : R^{p,q} \to \mathbb{R}^{p+1,q+1}$ are injective, hence ι becomes an embedding. From the definition of ι we get $\iota(\mathbb{R}^{p,q}) = \mathbb{P}(C \cup \{x \in C \mid \langle x, f_0\rangle_{p+1,q+1} \neq 0\}$, hence open and dense in $Q^{p,q}$. It remains to check that $\langle\cdot,\cdot\rangle_{p,q} \in \iota^* c$. Note first that $\iota^* c = [(\mu \circ \iota)^* \langle\cdot,\cdot\rangle_{p+1,q+1}]$ for a section $\mu : \mathbb{P}C \to C$. Specializing the section $\mu : \iota(U) \subset Q^{p,q} \to C^+$ for an open set $U \subset \mathbb{R}^n$ by $\mu(\iota(x)) := \iota(x)$ for all $y \in U$ yields

$(\mu \circ \iota)^* \langle \cdot, \cdot \rangle_{p+1,q+1} = 4\langle \cdot, \cdot \rangle_{p,q}$. This completes the proof, since we have shown that $\mathbb{R}^{p,q}$ has the Möbius sphere as conformal compactification. □

The next proposition deals with the group $\mathrm{Conf}(Q^{p,q})$ of conformal transformations of the Möbius sphere, in particular, we derive the group $\mathrm{Conf}(\mathbb{R}^{p,q})$ of conformal transformations of $\mathbb{R}^{p,q}$, see [Sch97, Theorem 2.6, Theorem 2.9].

Proposition 3.8 *For $\Lambda \in O(p+1, q+1)$ consider the map $\varphi_\Lambda : Q^{p,q} \to Q^{p,q}$ given by $\varphi_\Lambda(\mathbb{R}x) := \mathbb{R}(\Lambda x)$. Then one has*

1. *The map $\varphi : O(p+1, q+1) \to \mathrm{Conf}(Q^{p,q})$ is surjective with $\ker(\varphi) = \{\pm 1\}$.*
2. *For any $\phi \in \mathrm{Conf}_{loc}(\mathbb{R}^{p,q})$, there exists a unique $\hat{\phi} \in \mathrm{Conf}(Q^{p,q})$, such that on the domain of ϕ it holds that $\phi = \hat{\phi}$.*
3. *The group of conformal transformation of $\mathbb{R}^{p,q}$ is given by $\left(\mathbb{R}^+ \times O(p,q)\right) \ltimes \mathbb{R}^n$.*

The Möbius sphere carries a structure of a homogeneous space, like the sphere does in the semi Riemannian case. To identify this structure, let us define the Möbius group $G := O(p+1, q+1)/\{\pm I\}$, which acts transitively on $Q^{p,q}$. Introducing the stabilizer

$$B := stab_{P_\infty}(G) := \{g \in G \mid \mathbb{R}f_0 \text{ is a fix point of } g\} \subset G$$

of $P_\infty := \mathbb{R}f_0$ yields a diffeomorphism $Q^{p,q} \simeq G/B$. The structure of the Möbius sphere being a homogeneous space tells us that the groups G and B will play a role in describing conformal geometry in general. In an explicit form, the group B is isomorphic, with respect to the projection $O(p+1, q+1) \to G$, to the following subgroup of $O(p+1, q+1)$:

$$B \simeq \left\{ Z(a, A, v) := \begin{pmatrix} a^{-1} & v^t & b \\ 0 & A & x \\ 0 & 0 & a \end{pmatrix} \middle| \begin{array}{c} a \in \mathbb{R}^+, v \in \mathbb{R}^{p,q}, A \in O(p,q) \\ x := -aAJ^{p,q}v \\ b := -\frac{1}{2}a\langle v, v\rangle_{p,q} \end{array} \right\}. \tag{3.4}$$

Note that we always identify B with this subgroup. The conformal group $CO(p,q)$ is contained in B. To see this, note

$$\begin{aligned} B_0 :=& \{Z(a, A, 0) \in B\} \simeq CO(p,q) \\ B_1 :=& \{Z(1, I_n, v) \in B\} \simeq \mathbb{R}^n, \end{aligned}$$

where B_1 is abelian and normal in B, hence $B/B_1 \simeq CO(p,q)$. All together yields a $B-$principal bundle $(G, \pi, Q^{p,q}, B)$. This bundle will be generalized in terms of the first prolongation of the conformal frame bundle for arbitrary conformal manifolds. Because of their importance, we will study these groups in more detail in the next section. Note, if one starts with the Lie group $SO_0(p+1, q+1)/\{\pm I\}$, the action on $Q^{p,q}$ becomes effective.

Now let us do the same for the Lie group $\tilde{G} := Spin_0(p+1, q+1)$ which will appear in context of conformal spin calculus. Remember the twofold covering $\lambda : \tilde{G} \to SO_0(p+$

$1, q+1)$. This covering induces an action of $\tilde{G}$ on $Q^{p,q}$ in a natural way: $\tilde{G} \times Q^{p,q} \ni (g, \mathbb{R}v) \mapsto g \cdot \mathbb{R}v := \mathbb{R}(\lambda(g)v) \in Q^{p,q}$. Furthermore, considering the stabilizer $\tilde{B} \subset \tilde{G}$ of $P_\infty \in Q^{p,q}$ leads to:

Lemma 3.9 *The Lie group $\tilde{G}$ act transitively on the Möbius sphere. In particular, one has $Q^{p,q} \simeq \tilde{G}/\tilde{B}$ and $\tilde{B} = \lambda^{-1}(B)$.*

Proof. Since the action of $SO(p+1, q+1)$ is transitive and effective on $Q^{p,q}$, it follows that for $v, w \in C$ with $v \neq w$ there exists an $A \in SO(p+1, q+1)$, with $Av = w$. Now picking an element $g \in \lambda^{-1}(A)$ yields $g \cdot \mathbb{R}v = \mathbb{R}(\lambda(A)v) = \mathbb{R}w$. We do not get an effective action due to the twofold covering λ. The last claim follows from the definition of the $\tilde{B}-$action on $Q^{p,q}$. □

Thus the Möbius sphere admits a principal $\tilde{B}-$bundle $(\tilde{G}, \pi, Q^{p,q}, \tilde{B})$.

3.4 Lie groups and Lie algebras in conformal geometry

We have seen that the Lie groups $O(p+1, q+1)$ and $Spin(p+1, q+1)$ together with their subgroups play an important role in describing conformal (spin) geometry. This section describes these Lie groups and their Lie algebras in view of the computations we will perform later.

Let $G = O(p+1, q+1)/\{\pm I\}$ and fix the standard basis $\{e_0, \ldots, e_{n+1}\}$ in $\mathbb{R}^{p+1,q+1}$. Then define a new basis by: $f_0 := \frac{1}{\sqrt{2}}(e_{n+1} - e_0)$, $f_i := e_i$ for $1 \leq i \leq n$ and $f_{n+1} := \frac{1}{\sqrt{2}}(e_{n+1} + e_0)$. In this basis, the matrix representation of the semi Riemannian metric $\langle \cdot, \cdot \rangle_{p+1,q+1}$ is given by

$$S := \begin{pmatrix} 0 & 0 & 1 \\ 0 & J^{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J^{p,q} := \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

Since the orthogonal group $O(p+1, q+1)$ is the group preserving the inner product $\langle \cdot, \cdot \rangle_{p+1,q+1}$, we have $O(p+1, q+1) = \{A \in Mat(n+2, \mathbb{R}) \mid A^t \circ S \circ A = S\}$. Note that this realization is based on the basis $\{f_a\}_{a=0}^{n+1}$. The stabilizer $B := Stab_{\mathbb{R}f_0}(G)$ of the isotropic line $\mathbb{R}f_0$ is isomorphic, under the projection $O(p+1, q+1) \to G$, to the following subgroup of $O(p+1, q+1)$:

$$B \simeq \left\{ Z(a, A, v) := \begin{pmatrix} a^{-1} & v^t & b \\ 0 & A & x \\ 0 & 0 & a \end{pmatrix} \middle| \begin{array}{c} a \in \mathbb{R}^+, v \in \mathbb{R}^{p,q}, A \in O(p,q) \\ x := -aAJ^{p,q}v \\ b := -\frac{1}{2}a\langle v, v \rangle_{p,q} \end{array} \right\}.$$

It carries a semi direct product structure $B = B_0 \ltimes_\rho B_1$, where

$$B_0 := \left\{ X(a,A) := Z(a,A,0) = \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \, a \in \mathbb{R}^+, A \in O(p,q) \right\},$$

$$B_1 := \left\{ Y(v) := Z(0,0,v) = \begin{pmatrix} 1 & v^t & -\frac{1}{2}\langle v,v\rangle_{p,q} \\ 0 & I_n & -J^{p,q}v \\ 0 & 0 & 1 \end{pmatrix} \middle| \, v \in \mathbb{R}^n \right\},$$

and $\rho : B_0 \times B_1 \to B_1$ is the conjugation map, i.e., $\rho(b_0)b_1 := b_0 b_1 b_0^{-1}$ for $b_0 \in B_0$ and $b_1 \in B_1$. From the Lie algebraical point of view, we will also need the Lie subgroup

$$B_{-1} := \{Y(w)^t \mid w \in \mathbb{R}^n\}$$

of $O(p+1,q+1)$. The subgroup B_0 can be interpreted as the conformal group $CO(p,q) \simeq \mathbb{R}^+ \times O(p,q)$.

The Lie algebra $\mathfrak{g} = \mathfrak{o}(p+1,q+1)$ is a $|1|-$graded Lie algebra, i.e., it decomposes into $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ such that $[\mathfrak{b}_i, \mathfrak{b}_j] \subset \mathfrak{b}_{i+j}$, where $\mathfrak{b}_j$ is the Lie algebra of B_j for $j = -1,0,1$. With respect to the basis $\{f_a\}$ we have

$$\mathfrak{b}_{-1} = \left\{ M(x,(0,0),0) := \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & -x^t J^{p,q} & 0 \end{pmatrix} \middle| \, x \in \mathbb{R}^n \right\} \simeq \mathbb{R}^n,$$

$$\mathfrak{b}_0 = \left\{ M(0,(a,A),0) := \begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \, a \in \mathbb{R}, A \in \mathfrak{o}(p,q) \right\} \simeq \mathfrak{co}(p,q),$$

$$\mathfrak{b}_1 = \left\{ M(0,(0,0),z) := \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & -J^{p,q}z^t \\ 0 & 0 & 0 \end{pmatrix} \middle| \, z \in (\mathbb{R}^n)^* \right\} \simeq (\mathbb{R}^n)^*,$$

and finally

$$\mathfrak{g} = \left\{ M(x,(A,a),z) := \begin{pmatrix} -a & z & 0 \\ x & A & -J^{p,q}z^t \\ 0 & -J^{p,q}x^t & a \end{pmatrix} \middle| \, \begin{matrix} x \in \mathbb{R}^n, z \in (\mathbb{R}^n)^* \\ a \in \mathbb{R}, A \in \mathfrak{o}(p,q) \end{matrix} \right\}.$$

The semi direct product structure of B carries over to its Lie algebra since the differential of the conjugation $\rho : B_0 \times B_1 \to B_1$ yields the adjoint representation $d\rho : \mathfrak{b}_0 \to \mathfrak{gl}(\mathfrak{b}_1)$. Thus $LA(B) = \mathfrak{b}_0 \ltimes_{d\rho} \mathfrak{b}_1$. Since the Lie algebra $\mathfrak{o}(p+1,q+1)$ is semi simple, its Killing form $B_{\mathfrak{o}}(X,Y) = ntr(X \circ Y)$ is non-degenerate. Furthermore, it satisfies the following

$ad-$invariance: $B_{\mathfrak{o}}(ad(X)Y, Z) = -B_{\mathfrak{o}}(Y, ad(X)Z)$ for all $X, Y, Z \in \mathfrak{o}(p+1, q+1)$. The mapping

$$\begin{aligned} \Psi : \mathfrak{b}_1 &\to (\mathfrak{b}_{-1})^* \\ z &\mapsto \big(\mathfrak{b}_{-1} \ni x \mapsto \Psi(z)x := \frac{1}{2n} B_{\mathfrak{o}}(x, z)\big) \end{aligned}$$

defines an $ad(\mathfrak{b}_0)-$equivariant isomorphism, because of

$$\begin{aligned} \Psi(ad(b_0)b_1)(x) &= -\frac{1}{2n} B_{\mathfrak{o}}(x, ad(b_0)b_1) = -\frac{1}{2n} B_{\mathfrak{o}}(ad(b_0)x, b_1) \\ &= -\Psi(b_1)(ad(b_0)x) = \big[ad^*(b_0)(\Psi(b_1))\big]x, \end{aligned}$$

by the definition of the dual representation $(ad^*(Z)L)Y = L(-ad(Z)Y)$ for $Y, Z \in \mathfrak{g}$ and $L \in \mathfrak{g}^*$. Thus we can identify $\mathfrak{b}_1$ with $(\mathfrak{b}_{-1})^*$. The adjoint representation $Ad : G \to Gl(\mathfrak{g})$ yields a representation

$$Ad(b_0)v = aAv \in \mathfrak{b}_{-1}, \quad Ad(b_0)v^t = a^{-1}v^tA^{-1} \in \mathfrak{b}_1$$

on $v \in \mathfrak{b}_{-1}$ and $v^t \in \mathfrak{b}_1$ for $b_0 := (a, A) \in B_0$. As $exp : \mathfrak{b}_1 \to B_1$ with $exp(v^t) = Z(0, 0, v) \in B_1$ for any $v^t \in \mathfrak{b}_1$ is bijective, we get

$$Ad(exp(v^t))(a, A) = (a, A) + [v^t, (a, A)]$$

for any $(a, A) \in B_0$.

Now let us restrict our attention to the Lie group $Spin(p+1, q+1)$. Everything done before with respect to the group $O(p+1, q+1)$, can also be done with respect to the group $SO(p+1, q+1)$. Hence, all the subgroups we have presented have their analogue as subgroups of $SO(p+1, q+1)$. We lift the subgroups B, B_{-1}, B_0 and B_1 by the covering $\lambda : Spin(p+1, q+1) \to SO(p+1, q+1)$ and denote their lifts by $\tilde{\cdot}$. Similar to B, defining $\tilde{\rho}(g_0)g_1 := g_0 \cdot g_1 \cdot g_0^{-1}$ for $g_0 \in \tilde{B}_0$ and $g_1 \in \tilde{B}_1$ leads to $\tilde{B} = \tilde{B}_0 \ltimes_{\tilde{\rho}} \tilde{B}_1$. Hence $\tilde{B}/\tilde{B}_1 \simeq \tilde{B}_0$, which can be identified with the conformal spin group $CSpin(p, q) := \mathbb{R}^+ \times Spin(p, q)$.

Finally, let us check some properties of $\lambda : Spin(p+1, q+1) \to SO(p+1, q+1)$ and its differential $\lambda_* : \mathfrak{spin}(p+1, q+1) \to \mathfrak{so}(p+1, q+1)$. Recall the $\cdot^\flat-$ and $\cdot^\natural-$action induced by the semi Riemannian manifold $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{p,q})$, i.e., for $v \in \mathbb{R}^n$ the element $v^\flat \in (\mathbb{R}^n)^*$ is given by $v^\flat(x) = \langle v, x\rangle_{p,q}$ for all $x \in \mathbb{R}^n$, and for $\eta \in (\mathbb{R}^n)^*$ the element $\eta^\natural \in \mathbb{R}^n$ is given by $\langle \eta^\natural, x\rangle_{p,q} = \eta(x)$ for all $x \in \mathbb{R}^n$.

Lemma 3.10 *Let $Y(v) \in B_1$ with $v \in \mathbb{R}^n$, and $X(a, I_n) \in B_0$ with $a \in \mathbb{R}^+$. Then one has*

$$\lambda\left(1 + \frac{1}{2}\bar{v} \cdot f_0\right) = \begin{pmatrix} 1 & \bar{v}^\flat & -\frac{1}{2}|\bar{v}|^2 \\ 0 & I_n & -\bar{v} \\ 0 & 0 & 1 \end{pmatrix} = Y(v),$$

$$\lambda\left(\sqrt{\frac{1}{1-4\bar{a}^2}}(1+\bar{a}\tilde{f})\right)=\begin{pmatrix}a^{-1}&0&0\\0&I_n&0\\0&0&a\end{pmatrix}=X(a,I_n),$$

where $\bar{v}:=J^{p,q}v$, $\bar{a}:=\frac{1}{2}\frac{a-1}{a+1}$ *and* $\tilde{f}:=f_0\cdot f_{n+1}-f_{n+1}f_0$. *Furthermore, if* $x\in\mathfrak{b}_{-1}\simeq\mathbb{R}^{p,q}$, $z\in\mathfrak{b}_1\simeq(\mathbb{R}^{p,q})^*$ *and* $(b,A)\in\mathfrak{b}_0\simeq\mathfrak{co}(p,q)$, *then one has*

$$\begin{aligned}(\lambda_*)^{-1}(x)&=-\frac{1}{2}x\cdot f_{n+1},\\(\lambda_*)^{-1}(z)&=\frac{1}{2}z\cdot f_0,\\(\lambda_*)^{-1}((b,A))&=(\lambda_*)^{-1}(A)+\frac{b}{4}\tilde{f}.\end{aligned}$$

Proof. Let us start to prove the properties on the Lie group level. In case of $Z(0,0,v)\in B_1$ we first invert $1+\frac{1}{2}v\cdot f_0$ to $1-\frac{1}{2}v\cdot f_0$, since f_0 is isotropic. This element has spinor norm 1, hence $1+v\cdot f_0\in Spin(p+1,q+1)$. For $\alpha f_0+X+\beta f_{n+1}\in\mathbb{R}^{p+1,q+1}$ we get

$$\begin{aligned}\lambda(1+\frac{1}{2}v\cdot f_0)(\alpha f_0+X+\beta f_{n+1})&=(1+\frac{1}{2}v\cdot f_0)\cdot(\alpha f_0+X+\beta\cdot f_{n+1})\cdot(1-\frac{1}{2}vf_0)\\&=(\alpha+\langle v,X\rangle_{p,q}-\frac{1}{2}|v|^2\beta)f_0+(X-\beta v)+\beta f_{n+1}.\end{aligned}$$

The element $1+\bar{a}\tilde{f}$ has spinor norm $1-4\bar{a}^2$ and thus $\sqrt{\frac{1}{1-4\bar{a}^2}}(1+\bar{a}\tilde{f})$ is an element of the spin group $Spin(p+1,q+1)$. The inverse of $1+\bar{a}\tilde{f}$ is $b+c\tilde{f}$ for $b:=\frac{1}{1-4\bar{a}^2}$ and $c:=-\frac{\bar{a}}{1-4\bar{a}^2}$. This gives

$$\begin{aligned}\lambda\left(\sqrt{\frac{1}{1-4\bar{a}^2}}(1+\bar{a}\tilde{f})\right)(\alpha f_0+X+\beta f_{n+1})=&\lambda(1+\bar{a}\tilde{f})(\alpha f_0+X+\beta f_{n+1})\\=&(1+\bar{a}\tilde{f})(\alpha f_0+X+\beta f_{n+1})(b+c\tilde{f})\\=&\frac{1-2\bar{a}}{1+2\bar{a}}\alpha f_0+X+\frac{1+2\bar{a}}{1-2\bar{a}}\beta f_{n+1}\\=&a^{-1}\alpha f_0+X+a\beta f_{n+1}.\end{aligned}$$

Now we come to the Lie algebra properties. Recall that for an orthonormal basis $\{e_a\}_{a=0}^{n+1}$ of $\mathbb{R}^{p+1,q+1}$, we have a basis $\{E_{ij}:=\varepsilon_ie_je_i^t-\varepsilon_je_ie_j^t\}_{i<j}$ of $\mathfrak{so}(p+1,q+1)$. Furthermore, the Lie algebra $\mathfrak{spin}(p+1,q+1)$ possesses an algebra basis $\{e_i\cdot e_j\}_{i<j}$. Then the differential λ_* satisfies $\lambda_*(e_i\cdot e_j)=2E_{ij}$ for $i<j$. Showing the first equation is equivalent to showing

$$\lambda_*(X\cdot f_{n+1})=-2x.$$

For $\mathbb{R}^n \ni x = \sum_i x^i e_i$ we have

$$\lambda_*(x \cdot f_{n+1}) = \frac{1}{\sqrt{2}} \sum_i x^i \lambda_*(e_i \cdot e_{n+1} + e_i \cdot e_0) = \frac{2}{\sqrt{2}} \sum_i x^i (E_{in+1} - E_{0i}).$$

From the equations

$$\begin{aligned}\sum_i x^i (E_{in+1}(f_0) - E_{0i}(f_0)) &= -\sqrt{2}x,\\ \sum_i x^i (E_{in+1}(e_j) - E_{0i}(e_j)) &= \sqrt{2} \sum_i x^i \varepsilon_i \delta_{ij} f_{n+1},\\ \sum_i x^i (E_{in+1}(f_{n+1}) - E_{0i}(f_{n+1})) &= 0\end{aligned}$$

we may derive $\lambda_*(x \cdot f_{n+1}) = -2x$. The statement $\lambda_*(z \cdot f_0) = 2z$ is proved in a similar way. Let us come to the last equation. The assertion is proved by

$$\begin{aligned}\lambda_*(\lambda_*^{-1}(A) + \frac{b}{4}(f_0 \cdot f_{n+1} - f_{n+1} \cdot f_0)) &= A + \frac{b}{4}\lambda_*(-2e_0 \cdot e_{n+1})\\ &= A - bE_{0n+1} = (b, A),\end{aligned}$$

using

$$E_{0n+1}(f_0) = f_0, \quad E_{0n+1}(e_i) = 0, \quad E_{0n+1}(f_{n+1}) = -f_{n+1}.$$

That completes the proof. $\square$

Remark 3.11 Let $(M, [g])$ be conformal spin manifold. In order to compare the $g-$metric representation of the standard tractor bundle and standard spin tractor bundle, arising in Chapter 5, with respect to metrics g and $\hat{g} := e^{2\sigma} g$ from the conformal class, it is reasonable to specialize the entries $v := -[s]^{-1} \operatorname{grad}^g(\sigma) \in \mathbb{R}^n$ and $a := e^\sigma \in \mathbb{R}^+$ of Lemma 3.10. This gives us $\bar{a} = \frac{1}{2}\tanh(\frac{\sigma}{2})$, $N(1 + \bar{a}\tilde{f}) = \cosh(\frac{\sigma}{2})^{-2}$, and thus

$$\begin{aligned}\lambda\left(1 - \frac{1}{2}\bar{v} \cdot f_0\right) &= \begin{pmatrix} 1 & -([s]^{-1}\operatorname{grad}^g(\sigma))^\flat & -\frac{1}{2}|[s]^{-1}\operatorname{grad}^g(\sigma)|^2 \\ 0 & I_n & [s]^{-1}\operatorname{grad}^g(\sigma) \\ 0 & 0 & 1 \end{pmatrix}\\ &= Y\big(-[s]^{-1}\operatorname{grad}^g(\sigma)\big) =: b_1,\\ \lambda\left(\cosh(\frac{\sigma}{2})(1 + \bar{a}\tilde{f})\right) &= \begin{pmatrix} e^{-\sigma} & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & e^\sigma \end{pmatrix} = X(e^\sigma, I_n) =: b_0^{-1}.\end{aligned}$$

The elements $\tilde{b}_0^{-1} := \cosh(\frac{\sigma}{2})(1 + \bar{a}\tilde{f})$ and $\tilde{b}_1 := 1 - \frac{1}{2}\bar{v} \cdot f_0$ are lifts of $(b_0)^{-1}$ and b_1 with respect to λ. These elements will appear in Lemma 5.5 and Lemma 5.10.

3.5 First prolongation of the conformal frame bundle

In contrast to semi Riemannian manifolds, which possess a unique Levi-Civita connection on the tangent bundle, for conformal structures there does not exist a unique covariant derivative on the tangent bundle. The goal of this section is to associate a distinguished Cartan geometry of certain type to a conformal manifold. The corresponding bundle will be the first prolongation of the conformal frame bundle associated to a conformal manifold. It is inspired by looking at the $B-$principal bundle $(G, \pi, Q^{p,q}, B)$ of the Möbius sphere. Details and proofs of statements not proven here can be found in [ČSS97a, ČSS97b], and [Feh05]. Note, that we will present the first prolongation in a geometric way, done in [Feh05], whereas [ČSS97b] defined the first prolongation in an algebraic way.

Let (M, c) be a conformal manifold. Recall the Lie group $G = O(p+1, q+1)/\{\pm I\}$ together with their subgrougs presented in the Section 3.4. We denote by

$$\Big(\mathcal{P}^0 := \{u = (u_1, \dots, u_n)|\, h \in c, u \in \mathcal{P}^h\}, \pi, M, CO(p,q)\Big)$$

the conformal frame bundle associated to (M, c). The isomorphism

$$\begin{aligned}(\mathcal{P}^0 \times_{(B_0, Ad)} \mathfrak{b}_{-1})_x &\to T_xM \\ [u, v] &\mapsto \sum v_i u_i,\end{aligned}$$

where $u_x = (u_1, \dots, u_n) \in \mathcal{P}^0$ and $v \in \mathfrak{b}_{-1} \simeq \mathbb{R}^n$, leads to the soldering form $\theta^{\mathcal{P}^0} : T\mathcal{P}^0 \to \mathfrak{b}_{-1}$ of $\mathcal{P}^0$ defined by $\theta^{\mathcal{P}^0}_u(X) := [u]^{-1} d\pi_u(X)$, where $X \in T_u\mathcal{P}^0$ for $u \in \mathcal{P}^0$ and $[u] : \mathfrak{b}_{-1} \to T_xM$ is the usual isomorphism.

Lemma 3.12 *The soldering form $\theta^{\mathcal{P}^0}$ is strictly horizontal and B_0-equivariant, i.e., $\ker(\theta^{\mathcal{P}^0}_u) = Tv_u\mathcal{P}^0$ and $(R_b)^*\theta^{\mathcal{P}^0} = Ad(b^{-1})\theta^{\mathcal{P}^0}$ for $b \in B_0$. Furthermore, for any $X, Y \in \mathfrak{b}_0$ denote with $\tilde{X}, \tilde{Y}$ their fundamental vector fields. Then one has*

$$d\theta^{\mathcal{P}^0}(\tilde{X}, \tilde{Y}) = 0, \qquad d\theta^{\mathcal{P}^0}(\tilde{X}, \cdot) = -ad(X) \circ \theta^{\mathcal{P}^0}.$$

Proof. We have that $X \in \ker(\theta^{\mathcal{P}^0}_u)$ if and only if $\theta^{\mathcal{P}^0}_u(X) = [u]^{-1} d\pi_u(X) = 0$. Since $[u]^{-1}$ is an isomorphism, we conclude that $\theta^{\mathcal{P}^0}$ is strictly horizontal. The B_0-equivariance follows from

$$\begin{aligned}(R_h^*\theta^{\mathcal{P}^0})_u(X) =& [u \cdot h] d\pi_{u \cdot h} \circ (dR_h)_u(X) \\ =& Ad(h^{-1}) \circ [u]^{-1} d(\pi \circ R_h)_u(X) = Ad(h^{-1})\theta^{\mathcal{P}^0}_u(X),\end{aligned}$$

where $h \in B_0$, $u \in \mathcal{P}^0$, $X \in T_u\mathcal{P}^0$ and we have used that $[u \cdot h]^{-1}(w) = Ad(h^{-1})[u]^{-1}(w)$. The last two statements follows from $\ker(\theta^{\mathcal{P}^0}) = Tv\mathcal{P}^0$ and from the B_0-equivariance of $\theta^{\mathcal{P}^0}$. $\square$

Since for any horizontal subspace $H \subset T_u\mathcal{P}^0$ at a point $u \in \mathcal{P}^0$, i.e., $T_u\mathcal{P}^0 = H \oplus Tv_u\mathcal{P}^0$, the restriction of the soldering form $\theta_u^{\mathcal{P}^0}$ to H yields an isomorphism, we may define the torsion of a horizontal subspace by

$$t(H)(v,w) := d\theta_u\left((\theta_{|H}^{\mathcal{P}^0})^{-1}(v), (\theta_{|H}^{\mathcal{P}^0})^{-1}(w)\right) \in \mathfrak{b}_{-1},$$

where $v, w \in \mathfrak{b}_{-1}$. A horizontal space $H \subset T_u\mathcal{P}^0$ is called torsion free if $t(H)$ vanishes completely.

This is enough to define the first prolongation of $\mathcal{P}^0$ to be the set

$$\mathcal{P}^1 := \left\{H_u \subset T_u\mathcal{P}^0 \middle| u \in \mathcal{P}^0, H_u \text{ is horizontal and torsion free}\right\}.$$

Let us define a $B-$action on $\mathcal{P}^1$ by

$$H \cdot b := \left\{dR_{b_0}\left((\theta_{|H}^{\mathcal{P}^0})^{-1}(Ad(b_0) \circ \theta^{\mathcal{P}^0}(X))\right) + [Z, \widetilde{\theta^{\mathcal{P}^0}(X)}]_{\mathfrak{g}}(u \cdot b_0) \middle| X \in H\right\} \subset T_{u \cdot b_0}\mathcal{P}^0,$$

where $H \in \mathcal{P}^1$ and $b = (b_0, \exp(Z)) \in B = B_0 \ltimes_\rho B_1$. This means that an action of the form

$$H \cdot b_0 = dR_{b_0}(H) \subset T_{u \cdot b_0}\mathcal{P}^0 \tag{3.5}$$

describes a transport of H from the point u into the point $b_0 \cdot u$. An action of the form

$$H \cdot \exp(Z) := \{X + [\widetilde{Z, \theta(X)}]_{\mathfrak{g}} \mid X \in H\} \tag{3.6}$$

is given by a rotation of H inside $T_u\mathcal{P}^0$. The obvious projection $\pi^1(H_u) := u$ and $\pi^0(H_u) := \pi(u)$ and the fact that the $B-$, B_1-action on $\mathcal{P}^1 \to M$, respectively on $\mathcal{P}^1 \to \mathcal{P}^0$, is free and transitive, [ČSS97b, Section 1.5] or [Feh05, Lemma 5.3, 5.4], gives us:

Proposition 3.13 *Let (M, c) be a conformal manifold. Then, $(\mathcal{P}^1, \pi^1, \mathcal{P}^0, B_1)$ and $(\mathcal{P}^1, \pi^0, M, B)$ are principal bundles.*

In analogy to the soldering form $\theta^{\mathcal{P}^0}$, we want to define a soldering form $\theta^{\mathcal{P}^1} : T\mathcal{P}^1 \to \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$. First let us observe that the map

$$\begin{aligned} \mathcal{P}^1 \times_{B_1} (\mathfrak{b}_{-1} \oplus \mathfrak{b}_0) &\to T\mathcal{P}^0 \\ [H, X_{-1} + X_0] &\mapsto (\theta_{|H}^{\mathcal{P}^0})^{-1}(X_{-1}) + \tilde{X}_0(u) \end{aligned}$$

is an isomorphism of vector bundles over $\mathcal{P}^0$, compare [Feh05, Satz 5.7]. Then we define the soldering form of $\mathcal{P}^1$ by

$$\theta_H^{\mathcal{P}^1}(\xi) := [H]^{-1}(d\pi_H^1(\xi)), \tag{3.7}$$

where $H \in \mathcal{P}^1$, $[H] : \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \to T_u\mathcal{P}^0$ is the isomorphism given above and $\xi \in T_H\mathcal{P}^1$. The soldering form $\theta^{\mathcal{P}^1}$ splits by the decomposition $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ into

$$\theta^{\mathcal{P}^1} = \mathrm{pr}_{\mathfrak{b}_{-1}} \theta^{\mathcal{P}^1} + \mathrm{pr}_{\mathfrak{b}_0} \theta^{\mathcal{P}^1} =: \theta_{-1} + \theta_0$$

and satisfies: $\theta_{-1} = (\pi^1)^*\theta^{\mathcal{P}^0}$, $\theta_0(\widetilde{X_0 + X_1}(H)) = X_0$ for $X_0 + X_1 \in \mathfrak{b}_0 \oplus \mathfrak{b}_1$, $\theta^{\mathcal{P}^1}$ is $B-$equivariant, i.e., $(R_b)^*\theta^{\mathcal{P}^1} = \mathrm{proj}_{\mathfrak{b}_{-1}\oplus\mathfrak{b}_0} Ad(b^{-1})\theta^{\mathcal{P}^1}$ for $b \in B$, and $\ker(\theta^{\mathcal{P}^1}) = Tv\mathcal{P}^1$ with respect to $(\mathcal{P}^1, \pi^1, \mathcal{P}^0, B_1)$, see [ČSS97b, Section 1.6] or [Feh05, Satz 5.8]. The form $\theta^{\mathcal{P}^1}$ only depends on the conformal structure and will serve as a building block for a distinguished Cartan connection on $\mathcal{P}^1$ of type (G, B).

3.5.1 Normal conformal Cartan connection

Now we want to find a distinguished Cartan connection on the first prolongation $\mathcal{P}^1$ of the conformal frame bundle $\mathcal{P}^0$ associated to a conformal structure. The distinction of a Cartan connection involves two restrictions on the Cartan connection, namely, to be admissible and to satisfy a certain curvature condition.

Recall the decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$. Since w takes value in $\mathfrak{g}$, we can split the Cartan connection into three parts $w = w_{-1} + w_0 + w_1$.

Definition 3.14 *A Cartan connection w of type (G, B) on the first prolongation $\mathcal{P}^1$ is called* admissible *if the components w_{-1} and w_0 satisfy $w_{-1} \oplus w_0 = \theta^{\mathcal{P}^1}$.*

Any B_0-equivariant section $\sigma : \mathcal{P}^0 \to \mathcal{P}^1$, i.e., $\sigma(u \cdot b_0) = \sigma(u) \cdot b_0$ for all $u \in \mathcal{P}^0$ and $b_0 \in B_0$, defines a unique admissible Cartan connection w of type (G, B) on $\mathcal{P}^1$ such that $w_1(d\sigma(T\mathcal{P}^0)) = 0$, see [ČSS97a, Section 3.7] or [Feh05, Satz 6.2]. For example, fixing a $g \in c$ and extending the connection form $A^g \in \Omega^1(\mathcal{P}^g, \mathfrak{so}(p, q))$ to a torsion free connection form $\gamma^g \in \Omega^1(\mathcal{P}^0, \mathfrak{b}_0)$, for its definition we refer to the proof of Lemma 5.5, yields a B_0-equivariant section

$$\begin{aligned} \sigma^g : \mathcal{P}^0 &\to \mathcal{P}^1 \\ u &\mapsto \sigma^g(u) := \ker(\gamma^g_u). \end{aligned} \tag{3.8}$$

In this case, the unique admissible Cartan connection of type (G, B) with $w_1(d\sigma(T\mathcal{P}^0)) = 0$, is denoted by w^{A^g}, hence, the set of admissible Cartan connections of type (G, B) is not empty. The difference of two admissible Cartan connections $w, \hat{w}$ on $\mathcal{P}^1$ of type (G, B) is given by $w - \hat{w} = \Gamma \circ \theta_{-1}$ for a function $\Gamma \in \mathcal{C}(\mathcal{P}^1, (\mathfrak{b}_{-1})^* \otimes \mathfrak{b}_1)$. Properties of Γ can be found in [ČSS97a, Section 3.10] or [Feh05, Satz 6.3]. In the literature, Γ is often called the deformation tensor of two admissible Cartan connections w and $\hat{w}$.

The torsion t^w of an admissible Cartan connection is defined by

$$\begin{aligned} t^w : \mathcal{P}^1 &\to \Lambda^2(\mathfrak{b}_{-1} \oplus \mathfrak{b}_0)^* \otimes (\mathfrak{b}_{-1} \oplus \mathfrak{b}_0) \\ H &\mapsto t^w_H \\ t^w_H((X_{-1} + X_0, Y_{-1} + Y_0)) &:= (d\theta^{\mathcal{P}^1})_H(w_H^{-1}(X_{-1} + X_0), w_H^{-1}(Y_{-1} + Y_0)). \end{aligned}$$

Note that the complete information about the torsion is determined by its component in $(\mathfrak{b}_{-1})^* \wedge (\mathfrak{b}_{-1})^* \otimes \mathfrak{b}_0$, [ČSS97b, Section 2.1] or [Feh05, Lemma 6.4]. Hence, we will consider the torsion t^w_H to be an element in $\Lambda^2(\mathfrak{b}_{-1})^* \otimes \mathfrak{b}_1$ for $H \in \mathcal{P}^1$. In order to define an additionally curvature condition of an admissible Cartan connection let us denote by $\partial : (\mathfrak{b}_{-1})^* \otimes \mathfrak{b}_0 \to \Lambda^2(\mathfrak{b}_{-1})^* \otimes \mathfrak{b}_0$ the differential and by $\partial^* : \Lambda^2(\mathfrak{b}_{-1})^* \otimes \mathfrak{b}_0 \to (\mathfrak{b}_{-1})^* \otimes \mathfrak{b}_0$ the co-differential in the Spencer cohomology of the 1−graduated Lie algebra $\mathfrak{g}$, see [Och70]. They are given for $\eta \in (\mathfrak{b}_{-1})^* \otimes \mathfrak{b}_0$, $\varphi \in \Lambda^2(\mathfrak{b}_{-1})^* \otimes \mathfrak{b}_0$ and $v_0, v_1, v \in \mathfrak{b}_{-1}$ by

$$\partial\eta(v_1, v_2) := [v_0, \eta(v_1)]_{\mathfrak{g}} - [v_1, \eta(v_1)]_{\mathfrak{g}},$$
$$\partial^*\varphi(v) := \sum_{i=1}^{n} [Z^i, \varphi(X_i, v)]_{\mathfrak{g}},$$

where $\{X_i\}$ is a basis of $\mathfrak{b}_{-1}$ and $\{Z_i\}$ is their dual with respect to the Killing form $B_{\mathfrak{g}}$. Now we are able to state the following:

Proposition 3.15 *On the first prolongation $\mathcal{P}^1$, there exists a unique Cartan connection w such that w is admissible and such that the torsion t^w of w lies in the kernel of ∂^*_w, i.e., $\partial^*_w t^w_H = 0$ for all $H \in \mathcal{P}^1$.*

A proof can be found in [ČSS97b, Section 2.3] or in [Feh05, Satz 6.4].

Definition 3.16 *Let (M, c) be a conformal manifold. The uniquely determined Cartan connection of type (G, B) on the first prolongation $\mathcal{P}^1$ given by Proposition 3.15 is called the normal conformal Cartan connection and is denoted with w^{nc}.*

Thus we have asigned to a conformal manifold (M, c) a B−principal bundle $\mathcal{P}^1$ together with a normal conformal Cartan connection w^{nc}, analogously to semi Riemannian manifolds, where the $O(p, q)$−principal bundle $\mathcal{P}^g$ possess a unique torsion free connection form A^g induced by the Levi-Civita connection.

3.5.2 Metric representation of the normal conformal Cartan connection

Now we want to derive a representation of the normal conformal Cartan connection in terms of data with respect to a metric $g \in c$. Let us choose a metric $g \in c$ in the conformal class. Consider the $O(p, q)$−principal bundle $\mathcal{P}^g$ equipped with the connection form A^g induced by the Levi-Civita connection. We extend the connection A^g to a torsion free connection on $\mathcal{P}^0$, which we denote with γ^g. As mentioned before, γ^g induces a B_0−equivariant section $\sigma^g : \mathcal{P}^0 \to \mathcal{P}^1$ and therefore an unique admissible Cartan connection w^{A^g} on $\mathcal{P}^1$ of type (G, B) with $\mathrm{pr}_{\mathfrak{b}_1} w^{A^g}(d\sigma^g(T\mathcal{P}^0)) = 0$.

Lemma 3.17 *The pull-back of w^{A^g} by the section σ^g, given in equation* (3.8)*, is given by*

$$(\sigma^g)^* w^{A^g} = \theta^{\mathcal{P}^0} + \gamma^g.$$

Proof. With respect to the direct sum $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ we have $w^{A^g} = \theta^{\mathcal{P}^1} + w_1$, since w^{A^g} is admissible. From $\theta^{\mathcal{P}^1} = \theta_{-1} + \theta_0$ and $\theta_{-1} = (\pi^1)^*\theta^{\mathcal{P}^0}$ it follows that $(\sigma^g)^*\theta_{-1} = \theta^{\mathcal{P}^0}$. For $X \in \mathfrak{b}_0$ let $\tilde{X}$ denote its fundamental vector field on $\mathcal{P}^0$, and for $Y \in \sigma^g(u) = \ker(\gamma^g_u)$ we have $\gamma^g_u(\tilde{X}(u) \oplus Y) = X$. The same is true for $(\sigma^g)^*\theta_0$, since

$$\begin{aligned}((\sigma^g)^*\theta_0)_u\,(\tilde{X}(u) \oplus Y) &= \mathrm{pr}_{\mathfrak{b}_0}[\sigma^g(u)]^{-1}(\tilde{X}(u) \oplus Y)\\ &= \mathrm{pr}_{\mathfrak{b}_0}[\theta^{\mathcal{P}^0}_u(\tilde{X}(u) \oplus Y) \oplus X_0] = X_0,\end{aligned}$$

where $X_0 \in \mathfrak{b}_0$ is determined by

$$\tilde{X}_0(u) = \tilde{X}(u) + Y - (\theta^{\mathcal{P}^0}_{|\sigma^g(u)})^{-1}\theta^{\mathcal{P}^0}_u(\tilde{X}(u) \oplus Y) = \tilde{X}(u),$$

which comes from the inverse $[\sigma^g(u)]^{-1}$. Hence, $(\sigma^g)^*\theta_0(\tilde{X}(u) \oplus Y) = X$. Since all the computations were done with respect to $T_u\mathcal{P}^0 = Tv_u\mathcal{P}^0 \oplus \ker(\gamma^g_u)$, we may conclude $(\sigma^g)^*\theta_0 = \gamma^g$, which proves the lemma. $\square$

Since w^{nc} is admissible, its difference to w^{A^g} is given by a $\Gamma \in \mathcal{C}(\mathcal{P}^1, (\mathfrak{b}_{-1})^* \otimes \mathfrak{b}_1)$ such that

$$w^{nc}_H(\xi) = w^{A^g}_H(\xi) + \Gamma(H) \circ \theta_{-1}(\xi), \tag{3.9}$$

for any $H \in \mathcal{P}^1$ and $\xi \in T_H\mathcal{P}^1$. The pull back of equation (3.9) by the section σ^g is given by, see [ČSS97a, Section 6.3] or [Feh05, Satz 6.9],

$$((\sigma^g)^*w^{nc})_u(X) = \theta^{\mathcal{P}^0}_u(X) + \gamma^g_u(X) - \sum_{i=1}^n P^g_x(d\pi_u(X), s_i) \cdot e_i^*, \tag{3.10}$$

where $\pi : \mathcal{P}^0 \to M$ is the projection map, $u \in \mathcal{P}^0$, $X \in T_u\mathcal{P}^0$, P^g_x is the Schouten tensor with respect to g at a point $x = \pi^0(u)$ and $\{e_i\}_{i=1}^n$ is an orthonormal basis in $\mathfrak{b}_{-1} \simeq \mathbb{R}^{p,q}$ with dual basis $\{e_i^*\}_{i=1}^n$ in $\mathfrak{b}_1 \simeq (\mathbb{R}^{p,q})^*$ such that $\{s_i := [u, e_i]\}_{i=1}^n$ is an orthonormal basis in T_xM. Note our sign convention for the Schouten tensor.

That the Schouten tensor shows up is a consequence of the normalization condition of $w^{A^g} + \Gamma \circ \theta_{-1}$, compare [Feh05, Satz 6.8].

Remark 3.18 The choice of $g \in c$ yields a $O(p,q) \to CO(p,q)-$reduction of the conformal frame bundle $\mathcal{P}^0$ to the orthonormal frame bundle $\mathcal{P}^g$. Let us denote the reduction map by ι. Let us denote the composition $\sigma^g \circ \iota$ with $\sigma^{A^g} : \mathcal{P}^g \to \mathcal{P}^1$. From equation (3.10) we may derive for $s \in \mathcal{P}^g$ and $Y \in T_s\mathcal{P}^g$ that

$$((\sigma^{A^g})^*w^{nc})_s(Y) = \theta^{\mathcal{P}^0}_{\iota(s)}(d\iota_s(Y)) + \gamma^g_{\iota(s)}(d\iota_s(Y)) - \sum_{i=1}^n P^g_x(d\pi^0_{\iota(s)}(d\iota_s(Y)), s_i) \cdot e_i^*$$

$$=[s]^{-1}(d\pi^g_s(Y)) + A^g_s(Y) - \sum_{i=1}^{n} P^g_x(d\pi^g_s(Y), s_i) \cdot e^*_i, \tag{3.11}$$

where $\pi^g : \mathcal{P}^g \to M$ is the projection map and $\{e_i\}_{i=1}^n$ is an orthonormal basis in $\mathfrak{b}_{-1} \simeq \mathbb{R}^{p,q}$ with dual basis $\{e^*_i\}_{i=1}^n$ in $\mathfrak{b}_1 \simeq (\mathbb{R}^{p,q})^*$ such that $\{s_i := [s, e_i]\}_{i=1}^n$ is an orthonormal basis in $T_{\pi(u)}M$.

3.6 First prolongation of the conformal spin structure

Let (M, c) be a conformal spin manifold. In spirit of the conformal frame bundle $\mathcal{P}^0$ of a conformal manifold (M, c) with structure group $B_0 \simeq CO_0(p, q) = R^+ \times SO_0(p, q)$, we consider the conformal spin group $CSpin_0(p, q) := R^+ \times Spin_0(p, q)$ and extend $\lambda : Spin_0(p, q) \to SO_0(p, q)$ trivially to $CSpin_0(p, q)$, i.e.,

$$\begin{aligned} \lambda^0 : CSpin_0(p, q) &\to CO_0(p, q) \\ (a, g) &\mapsto \lambda^0 ((a, g)) := (a, \lambda(g)) . \end{aligned}$$

Definition 3.19 *Let (M, c) be a conformal manifold. A* conformal spin structure *on (M, c) is a $\lambda^0 : CSpin_0(p, q) \to SO_0(p, q)-$reduction $(\mathcal{Q}^0, f^0)$ of the conformal frame bundle $\mathcal{P}^0$, i.e., $\mathcal{Q}^0$ is an $CSpin_0(p, q)-$principal bundle with a smooth map $f^0 : \mathcal{Q}^0 \to \mathcal{P}^0$ such that $\pi_{\mathcal{P}^0} \circ f^0 = \pi_{\mathcal{Q}^0}$ and $f^0 ((a, g) \cdot q) = \lambda^0 ((a, g)) \cdot f^0(q)$ for all $(a, g) \in CSpin_0(p, q)$ and $q \in \mathcal{Q}^0$.*

We call (M, c) a conformal spin manifold *if (M, c) admits a conformal spin structure.*

Lemma 3.20 *$(M, c = [g])$ is a conformal spin manifold if and only if (M, g) is a spin manifold.*

Proof. Let $(\mathcal{Q}^0, f^0)$ be a conformal spin structure on $(M, [g])$. The conformal frame bundle $\mathcal{P}^0$ reduces to $\mathcal{P}^g$, i.e., $\iota : \mathcal{P}^g \to \mathcal{P}^0$ is a $SO_0(p, q) \to CO_0(p, q)-$reduction, and we define a spin structure on (M, g) by

$$\mathcal{Q}^g := \{q \in \mathcal{Q}^0 \mid f^0(q) \in \iota(\mathcal{P}^g)\}$$

and $f^g := f^0_{|\mathcal{Q}^g} : \mathcal{Q}^g \to \mathcal{P}^g$. Conversely, due to the reduction property we have $\mathcal{P}^0 \simeq \mathcal{P}^g \times_{SO_0(p,q)} CO_0(p, q)$, hence a spin structure $(\mathcal{Q}^g, f^g)$ on (M, g) defines a conformal spin structure by enlargements, $\mathcal{Q}^0 := \mathcal{Q}^g \times_{Spin_0(p,q)} CSpin_0(p, q)$ and $f^0 := f^g \times \lambda^0$. $\square$

Note that in what follows we will also denote by λ the twofold covering of $Spin_0(p + 1, q + 1)$ to $SO_0(p + 1, q + 1)$. Let (M, c) be a conformal spin manifold, and let $(\mathcal{Q}^0, f^0)$ be a conformal spin structure on (M, c). Recall the Lie group $\tilde{G} = Spin_0(p + 1, q + 1)$ together with their subgroups $\tilde{B}$, $\tilde{B}_0$ and $\tilde{B}_1$, introduced in Section 3.4. The goal is now to define the first prolongation of a conformal spin structure on (M, c). Essentially, it will be induced by the first prolongation $\mathcal{P}^1$ of the conformal frame bundle. The first prolongation of a conformal spin structure $(\mathcal{Q}^0, f^0)$ on (M, c) is define by

$$\mathcal{Q}^1 := \{\tilde{H}_q \subset T_q\mathcal{Q}^0 \mid q \in \mathcal{Q}^0, df^0_q(\tilde{H}) \in \mathcal{P}^1\}.$$

Furthermore, we define a $\tilde{B}-$action on $\mathcal{Q}^1$ by

$$\tilde{H}_q \cdot \tilde{b} := (df^0_{q\cdot b_0})^{-1}\left(df^0_q(\tilde{H}) \cdot \lambda(\tilde{b})\right) \subset T_{q\cdot\tilde{b}_0}\mathcal{Q}^0,$$

where $\tilde{H}_q \in \mathcal{Q}^1$ and $\tilde{b} = \tilde{b}_0 \cdot \tilde{b}_1 \in \tilde{B}_0 \ltimes_{\tilde{\rho}} \tilde{B}_1$. Introducing the projections $\tilde{\pi}^1(\tilde{H}_q) = q$ and $\tilde{\pi}^0(\tilde{H}_q) := \pi(q)$, and the map $f^1 := df^0 : \mathcal{Q}^1 \to \mathcal{P}^1$ yields

Proposition 3.21 *Let (M, c) be a conformal spin manifold. Then, the bundles $(\mathcal{Q}^1, \tilde{\pi}^1, \mathcal{Q}^0, \tilde{B}_1)$ and $(\mathcal{Q}^1, \tilde{\pi}^0, M, \tilde{B})$ are principal bundles. Furthermore, $(\mathcal{Q}^1, f^1)$ is a $\lambda : Spin_0(p+1, q+1) \to SO_0(p+1, q+1)-$reduction of the first prolongation $\mathcal{P}^1$ of the conformal frame bundle.*

Any Cartan connection w of $\mathcal{P}^1$ induces by $\tilde{w} := \lambda_*^{-1} \circ w \circ df^1$ a Cartan connection of type $(Spin_0(p+1, q+1), \tilde{B})$ on $\mathcal{Q}^1$. In particular, taking the normal conformal Cartan connection w^{nc} leads to the normal conformal spin connection

$$\tilde{w}^{nc} \in \Omega^1\left(\mathcal{Q}^1, \mathfrak{spin}(p+1, q+1)\right).$$

Having the triple $(\mathcal{Q}^1, f^1, \tilde{w}^{nc})$, a choice $g \in c$ defines a $\tilde{B}_0-$ equivariant map

$$\begin{aligned} \tilde{\sigma}^g : \mathcal{Q}^0 &\to \mathcal{Q}^1 \\ q &\mapsto \tilde{\sigma}^g(q) := \ker(\tilde{\gamma}^g_q) \in \mathcal{Q}^1, \end{aligned}$$

where $\tilde{\gamma}^g$ is the extension of $\tilde{A}^g \in \Omega^1(\mathcal{Q}^g, \mathfrak{spin}(p, q))$ to the conformal spin structure $\mathcal{Q}^0$. Note that $\tilde{\sigma}^g$ makes the resulting diagramm commutative, i.e., $f^1 \circ \tilde{\sigma}^g = \sigma^g \circ f$. The pull back of the normal conformal spin connection with respect to $\tilde{\sigma}^g$ is given for $q \in \mathcal{Q}^0$ and $\tilde{X} \in T_q\mathcal{Q}^0$ by

$$\left((\tilde{\sigma}^g)^*\tilde{w}^{nc}\right)_q(\tilde{X}) = \lambda_*^{-1}[\theta^{\mathcal{P}^0}_{f^0(q)}(X) + \gamma^g_{f^0(q)}(X) - \sum_i P^g_x(d\pi_{f^0(q)}(X), s_i)e_i^*], \tag{3.12}$$

where $X := df^0_q(\tilde{X}) \in T_{f^0(q)}\mathcal{P}^0$, $\pi : \mathcal{P}^0 \to M$ is the projection map, x is the base point of q, $\{e_i\}$ is an orthonormal basis in $\mathfrak{b}_{-1}$ and $\{e_i^*\}$ denotes their dual basis in $\mathfrak{b}_1$ such that $\{s_i := [f^0(q), e_i]\}$ is an orthonormal basis of T_xM.

Remark 3.22 The choice of $g \in c$ yields a $Spin_0(p, q) \to CSpin_0(p, q)-$reduction of the conformal spin structure $(\mathcal{Q}^0, f^0)$ to the spin structure $(\mathcal{Q}^g, f^g)$. Let us denote the reduction map by $\tilde{\iota}$ and the composition $\tilde{\sigma}^g \circ \tilde{\iota}$ by $\tilde{\sigma}^{A^g} : \mathcal{Q}^g \to \mathcal{Q}^1$. Note that the reduction maps $\iota : \mathcal{P}^g \to \mathcal{P}^0$ and $\tilde{\iota} : \mathcal{Q}^g \to \mathcal{Q}^1$ satisfy $f^0 \circ \tilde{\iota} = \iota \circ f^g$. From equation

(3.12) we may derive for $q \in \mathcal{Q}^g$ and $\tilde{Y} \in T_q\mathcal{Q}^g$ that

$$\left((\tilde{\sigma}^{A^g})^*\tilde{w}^{nc}\right)_q(\tilde{Y}) = \lambda_*^{-1}\left([f^g(q)]^{-1}d\pi^g_{f^g(q)}(Y) + A^g_{f^g(q)}(Y) - \sum_i P^g_x(d\pi^g_{f^g(q)}(Y), s_i)e_i^*\right), \tag{3.13}$$

where $\pi^g : \mathcal{P}^g \to M$ is the projection map, $Y := df^g_q(\tilde{Y}) \in T_{f^g(q)}\mathcal{P}^g$, $x := \pi^g \circ f^g(q)$, $\{e_i\}$ is an orthonormal basis in $\mathfrak{b}_{-1}$ and $\{e_i^*\}$ denotes their dual basis in $\mathfrak{b}_1$ such that $\{s_i := [f^g(q), e_i]\}$ is an orthonormal basis in T_xM.

3.7 The Fefferman-Graham construction for conformal manifolds

Another way to derive a conformal calculus is given by the ambient metric construction, or, equivalently the Poincaré metric introduced by Fefferman and Graham [FG85, FG07, FG11].

Let $(M^{p,q}, c)$ be a conformal manifold of dimension $n = p+q$. Define a subbundle $Q := \cup_{x\in M}\{g_x \mid \lambda \in \mathbb{R}^+, g \in c\} \subset S^2(T^*M)$ called the metric bundle of (M, c). A choice $g \in c$ trivializes the bundle $Q \simeq \mathbb{R}^+ \times M$ by $h_x = t^2 g_x \mapsto (t, x)$. In terms of another $\hat{g} = e^{2\sigma}g$ we have $\hat{t} = e^{-\sigma}t$. There is a natural $\mathbb{R}^+-$action on Q given by $\delta_s(\lambda^2 g_x) := (s\lambda)^2 g_x$. Thus $(Q, \pi, M, \mathbb{R}^+)$ is a $\mathbb{R}^+-$principal bundle over M whose sections $\sigma : M \to Q$ represent elements in the conformal structure c of M. One can define a symmetric bilinear form g_0 on the metric bundle Q by $(g_0)_{g_x}(X, Y) := g_x(d\pi_{g_x}(X), d\pi_{g_x}Y)$ for $X, Y \in T_{g_x}Q$. The form g_0 degenerates in direction of the vertical tangent space $Tv_{g_x}Q = \ker(d\pi_{g_x}) = span_{\mathbb{R}}(T(x))$, where $T = \frac{d}{ds}_{|s=1}\delta_s$ is the infinitesimal generator of δ. By the definition of g_0 we have $\delta_s^*(g_0) = s^2 g_0$, i.e., g_0 is homogeneous of degree 2. The space we want to work with is given by the enlargement of Q to $\tilde{Q} := Q \times (-1, 1)$ and the new coordinate will be denoted by ρ. It is convenient to choose the embedding $\iota(z) := (z, 0)$ of Q into $\tilde{Q}$. We extend trivially the $\mathbb{R}^+-$action and the infinitesimal generator to $\tilde{Q}$ and still denote them by δ and T. Having all the notation fixed, we can make the following definition:

Definition 3.23 *Let $(M^{p,q}, [h])$ be a conformal manifold with $p+q \geq 3$. An ambient space of $(M, [h])$ is a pseudo Riemannian manifold $(\tilde{Q}, \tilde{g})$ of signature $(p+1, q+1)$ such that:*

(1) The embedding $\iota : Q \to \tilde{Q}$ satiesfies $\iota^*\tilde{g} = g_0$.

(2) The $\mathbb{R}^+-$action $\tilde{\delta}$ on $\tilde{Q}$ fullfills $\tilde{\delta}_s^*\tilde{g} = s^2\tilde{g}$ and $\iota \circ \delta_s = \tilde{\delta}_s \circ \iota$.

(3) If M is odd dimensional, then $Ric(\tilde{g}) = O(\rho^\infty)$.

(3') If M is even dimensional, then $Ric(\tilde{g}) = O^+(\rho^{\frac{n}{2}-1})$, i.e., $Ric(\tilde{g}) = O(\rho^{\frac{n}{2}-1})$, and $Ric(\tilde{g})_{00}$, $Ric(\tilde{g})_{0i}$ and $\sum_{i,j=1}^n h^{ij}Ric(\tilde{g})_{ij}$ are $O(\rho^{\frac{n}{2}})$ for $i = 1, \dots n$, respectively.

Two ambient spaces $(\tilde{Q}_1, \tilde{g}_1)$ and $(\tilde{Q}_2, \tilde{g}_2)$ are called ambient equivalent *if there exist open sets $U_1 \subset \tilde{Q}_1$ and $U_2 \subset \tilde{Q}_2$ and a diffeomorphism $\Phi : U_1 \to U_2$, with the following properties: Both sets U_1 and U_2 contain M; The diffeomorphims Φ, when restricted to M, is the identity; U_i, for $i = 1, 2$, are dilation-invariant, and Φ commutes with dilations; If n is odd, then $\tilde{g}_1 - \Phi^* \tilde{g}_2 = O(\rho^\infty)$, and, if n is even, then $\tilde{g}_1 - \Phi^* \tilde{g}_2 = O^+(\rho^{\frac{n}{2}})$, i.e., $\tilde{g}_1 - \Phi^* \tilde{g}_2 = O(\rho^{\frac{n}{2}})$, and $(\tilde{g}_1 - \Phi^* \tilde{g}_2)_{00}$, $(\tilde{g}_1 - \Phi^* \tilde{g}_2)_{0i}$ and $g^{ij}(\tilde{g}_1 - \Phi^* \tilde{g}_2)_{ij}$ are $O^+(\rho^{\frac{n}{2}+1})$ for $i, j = 1, \ldots n$, respectively.*

The technicality of this definition unsures us to present the following Proposition, compare [FG07, Theorem 2.3] or [FG11, Chapters 2 and 3].

Proposition 3.24 *Let $(M, [h])$ be a conformal manifold of dimension $n \geq 2$. Then there exist, up to ambient equivalence, a unique formal ambient space $(\tilde{Q}, \tilde{g})$ of (M, c).*

The term formal in the previous proposition indicates that the metric $\tilde{g}$ is a formal power series at $\rho = 0$.

Remark 3.25 Let us briefly recall the structure of the metric $\tilde{g}$ of the proposition above. Let us choose a metric $h \in [h]$ and suitable coordinates (t, x, ρ) on $\tilde{Q}$, such that we have locally

$$\tilde{g}_{(t,x,\rho)} = t^2 \sum_{i,j=1}^{n} \tilde{g}_{ij}(x, \rho) dx_i dx_j + 2t dt d\rho + e(x, \rho) dt dt + \sum_{k=1}^{n} t d_k(x, \rho) dt dx_k$$

for functions $\tilde{g}_{ij}(x, \rho)$, $e(x, \rho)$, $d_k(x, \rho)$ satisfying $\tilde{g}_{ij}(x, 0) = h_{ij}(x)$, $e(x, \rho) = 0$ and $d_k(x, 0) = 0$. Now the assumption of $\tilde{g}$ to be Ricci flat fixes the unknown functions e, $\tilde{g}_{ij}$ and d_k. This can be done by taking formal power series with respect to $\rho = 0$ for e, $\tilde{g}_{ij}$ and d_k and calculating the unknown coefficents successively from the Ricci flatness condition. This results in

$$\tilde{g}_{(t,x,\rho)} = t^2 \sum_{i,j=1}^{n} \tilde{g}_{ij}(x, \rho) dx_i dx_j + 2t dt d\rho + 2\rho dt dt. \tag{3.14}$$

In case M is odd dimensional, the formal power series $\tilde{g}_{ij}(x, \rho) = h_{ij}(x) + 2\rho P_{ij}(x) + \ldots$ is completly determined by h. Here P denotes the Schouten tensor with respect to h. On the other hand, in case M is even dimensional, the determination of the unknown coefficients is obstructed at a certain order of ρ by the Ricci flatness condition. Thus one can only determine the coefficients of $\tilde{g}_{ij}$ up to order $\frac{n}{2} - 1$, i.e., $\tilde{g}_{ij}(x, \rho) = h_{ij}(x) + 2\rho P_{ij}(x) + \ldots + \rho^{\frac{n}{2}-1} \tilde{g}_{ij}^{(\frac{n}{2}-1)}(x) + O(\rho^{\frac{n}{2}})$. Again, all the coefficients of $\tilde{g}_{ij}$ are given in terms of h, as long as they are not obstructed. This reflects the technical definition of the ambient metric in even dimension.

Example 3.26 Consider the sphere $(S^n, [g_c])$ equipped with the conformal class induced by the round metric $g_c := (\langle \cdot, \cdot \rangle_{n+1})_{|S^n}$. The ambient space for $(S^n, [g_c])$ is the

Minkowski space $(\mathbb{R}^{n+2}, \langle\cdot,\cdot\rangle_{1,n+1})$. To see that, consider the positive light cone C^+ on $\mathbb{R}^{1,n+1}$. On $\mathbb{R}^{n+2}$, we have a natural $\mathbb{R}^+-$action δ given by $\delta_s(x) := sx$ for $s \in \mathbb{R}^+$ and $x \in \mathbb{R}^{n+2}$ which also restricts to C^+. Thus, it is clear that the light cone is a $\mathbb{R}^+-$principal bundle over its projectivization $\mathbb{P}C^+$, i.e., $(C^+, \pi, \mathbb{P}C^+, \mathbb{R}^+)$. Let us denote by g_0 the pull back of $\langle\cdot,\cdot\rangle_{1,n+1}$ to a $(0,2)-$tensor on C^+ by the inclusion $C^+ \subset \mathbb{R}^{n+2}$, which is symmetric. Along the vertical tangent space $TvC^+ := \ker(d\pi) \subset TC^+$ the form g_0 is degenerate. Recall that the projective light cone possesses a conformal structure induced by $\langle\cdot,\cdot\rangle_{1,n+1}$ and sections $\mu : \mathbb{P}C^+ \to C^+$. Since we can identify $(\mathbb{P}C^+, [\mu^*\langle\cdot,\cdot\rangle_{1,n+1}])$ with $(S^n, [g_c])$ isometrically by the mapping $\mathbb{P}C^+ \ni [x_0 : \ldots : x_n] \mapsto (1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \in S^n$ we found the ambient space $(\mathbb{R}^{1,n+1}, \langle\cdot,\cdot\rangle_{1,n+1})$ for the conformal manifold $(S^n, [g_c])$.

Further examples of ambient spaces for locally conformally flat structures and conformally Einstein structures are given in [FG07, Section 7] or [FG11, Chapter 7].

Another way to encode a conformal structure $(M, [h])$ is given by the Poincaré metric. Consider a manifold X^+ with boundary $M := \partial X^+$. Let $r : X^+ \to \mathbb{R}^+_0$ denote a defining function for the boundary, i.e., $\{r = 0\} = M$ and $dr_{|TM} \neq 0$. A semi Riemannian metric g_+ on $int(X^+)$ is called conformally complete, if $r^2 g_+$ extends to a metric on X^+ and $(r^2 g_+)_{|M}$ is nondegenerate. Is additionaly $(r^2 g_+)_{|M} \in c$ fulfilled we call g_+ conformally complete with conformal infinity $(M, [h])$.

Definition 3.27 *A Poincaré metric for $(M^n, [h])$ is a conformally complete metric g_+, such that:*

(1) g_+ has conformal infinity (M, c).

(2) In case of n odd, then $Ric(g_+) + ng_+ = O(r^\infty)$.

(2') In case of n even, $Ric(g_+)+ng_+ = O^+(r^{n-2})$, i.e., $Ric(g_+)+ng_+ = O(r^{n-2})$, and $Ric(g_+)_{0i} + n(g_+)_{0i}$, $Ric(g_+)_{ij} + n(g_+)_{ij}$ and $\sum_{i,j=1}^n h^{ij}(Ric(g_+)_{ij} + n(g_+)_{ij})$ are $O(r^{n-1})$ for $i = 1, \ldots, n$, respectively.

Two Poincaré metrics $(X_1^+, (g_+)_1)$ and $(X_2^+, (g_+)_2)$ are called Poincaré equivalent if there exist open sets $U_1 \subset X_1^+$ and $U_2 \subset X_2^+$ and a diffeomorphism $\Phi : U_1 \to U_2$, with the following properties: Both sets U_1 and U_2 contain M; The diffeomorphism Φ, when restricted to M, is the identity; If n is odd, then $(g_+)_1 - \Phi^*(g_+)_2 = O(r^\infty)$, and, if n is even, then $(g_+)_1 - \Phi^*(g_+)_2 = O^+(r^{n-2})$, i.e., $(g_+)_1 - \Phi^*(g_+)_2 = O(r^{n-2})$, and $((g_+)_1 - \Phi^*(g_+)_2)_{00}$, $((g_+)_1 - \Phi^*(g_+)_2)_{0i}$ and $\sum_{i,j=1}^n h^{ij}\left((g_+)_1 - \Phi^*(g_+)_2\right)_{ij}$ are $O(r^{n-1})$ for $i = 1, \ldots, n$, respectively.

Then the following is proven in [FG07, Theorem 4.4] or [FG11, Chapter 4] .

Proposition 3.28 *Let $(M, [h])$ be a conformal manifold. Up to Poincaré equivalence, there exists a unique formal Poincaré metric (X^+, g_+) for $(M, [h])$.*

Remark 3.29 Let $(M, [h])$ be a conformal manifold and denote by g_+ its Poincaré metric modeled on X^+. The representative $h \in [h]$ induces a unique defining function r

of the boundary $\partial X^+ = M$, such that locally the Poincaré metric takes the form

$$g_+ = r^{-2}(dr^2 + h_r), \tag{3.15}$$

where h_r is a $1-$parameter group of metrics on M. The proof of Proposition 3.28 is constructive, so we find an even formal power series $h_r = h - r^2P + r^4h^{(4)} + \ldots$ in case M is odd dimensional. Again the Schouten tensor P with respect to h has shown up. In case M is even dimensional, the determination of the coefficents of h_r are obstructed at order n, thus $h_r = h - 2r^2P + \ldots + r^{n-2}(h)^{(n-2)} + r^n(h)_0^{(n)} + O(r^n)$, where $h_0^{(n)}$ is trace free part of $h^{(n)}$. To end up we mention that the fourth coefficient of h_r is given by $\frac{1}{4(n-4)}((n-4)P^2 - B)$, where B denotes the Bach tensor associated to h.

Example 3.30 Again, consider the conformal manifold $(S^n, [g_c])$, where g_c is the round metric. In this case, the Poincaré metric is modeled on the Ball $B^{n+1} := \{x \in \mathbb{R}^{n+1} \mid |x| < 1\}$ equipped with the hyperbolic metric $g_h = \frac{4}{(1-|x|^2)^2}\langle\cdot,\cdot\rangle_{0,n+1}$. In spherical coordinates $x = (x_1, \ldots, x_{n+1}) \mapsto (l, w)$ with

$$l := |x|, \text{ and } w := \frac{x}{|x|}$$

the metric $\langle\cdot,\cdot\rangle_{0,n+1}$ takes the form $dl^2 + l^2g_c$, where g_c is the round metric on $S^n \subset \mathbb{R}^{n+1}$. Now let us choose $h = \frac{1}{4}g_c$ and take the defining function $r(x) := \frac{1-|x|}{1+|x|}$ for $x \in \mathbb{R}^{n+1}$ of the sphere S^n. Thus the hyperbolic metric on the upper half of B^{n+1} has the local representation $g_h = r^{-2}(dr^2 + \frac{1-r^2}{4}g_c)$ with respect to the coordinates (r, w). Thus the space (B^{n+1}_+, g_h) is the Poincaré model for the standard model in conformal geometry.

According to [FG07, Propositions 4.6 and 4.7], or [FG11, Chapter 4], the construction of formal ambient metrics and formal Poincaré metrics are equivalent.

3.8 Conformally covariant differential operators

In this section, we will define the (infinitesimal) conformal covariance of certain bi-degree of a geometric differential operator acting on the spinor bundle of a spin manifold. Furthermore, we will provide an equivalence between conformal covariance and infinitesimal conformal covariance. This characterization formulated for tensors can be found in [GW86], whereas for geometric differential operators acting on mixed tensor bundles see [Bra85] or [Wün86].

Let (M, g) be a spin manifold. Although the spinor bundle $S(M, g)$ depends on the metric g, there is a way of identifying spinor bundles arising from conformally equivalent metrics, see Section 3.2. Consider $g, \hat{g} = e^{2\sigma}g \in c$ two representatives of the conformal class. The identification of their spinor bundles was denoted by $F_\sigma : S(M, g) \to S(M, \hat{g})$. Now consider a geometric differential operator $D(g) : \Gamma\big(S(M, g)\big) \to \Gamma\big(S(M, g)\big)$, i.e.,

$$D(g) = \text{polynomial}(\nabla^{S(M,g)}, \mathcal{R}\otimes, g\otimes, g^{-1}\otimes, \text{contractions}, \mu).$$

Note that Clifford multiplication μ can also appear in the expression of $D(g)$.

A geometric differential operator $D(g)$ has conformal weight $w \in \mathbb{R}$ if under uniform dilation of the metric, i.e., $\sigma = const$, it holds that $F_\sigma^{-1} \circ D(e^{2\sigma} g) \circ F_\sigma = e^{w\sigma} D(g)$.

Example 3.31 Clifford multiplication of a spinor with a vector field has conformal weight $w = +1$, since $F_\sigma^{-1}(X \hat{\cdot} F_\sigma(\psi)) = e^\sigma X \cdot \psi$. The covariant derivative $\nabla^{S(M,g)}$ has conformal weight $w = 0$, due to $F_\sigma^{-1}(\nabla_X^{S(M,\hat{g})} F_\sigma(\psi)) = \nabla_X^{S(M,g)} \psi$, see equation (3.2). Finally, the Dirac operator is constructed from the covariant derivative together with Clifford multiplication. It satisfies $F_\sigma^{-1} \circ D\!\!\!\!/(e^{2\sigma}) \circ F_\sigma = e^{-\sigma} D\!\!\!\!/(g)\psi$ and thus it has conformal weight $w = -1$.

Let $D(g)$ be a geometric differential operator of conformal weight w. For any $a \in \mathbb{R}$ and $g, \hat{g} = e^{2\sigma} g \in c$ we have

$$D(e^{2\sigma} g)(e^{a\sigma} F_\sigma \psi) = e^{b\sigma} F_\sigma [D(g)\psi + R_a(g;\sigma)\psi] \tag{3.16}$$

for $b := a + w$ and for a geometric differential operator $R_a(g;\sigma)$, which also depends on σ and derivatives thereof. Observe that $R_a(g;\sigma)$ collects all the extra terms arising from differentiations of the conformal factor σ. Thus one can decompose $R_a(g;\sigma)$ into

$$R_a(g;\sigma) = R_a^1(g;\sigma) + \ldots + R_a^l(g;\sigma), \tag{3.17}$$

where $R_a^i(g;\sigma)$, for i less than or equal to the maximal power of differentiations inside $D(g)$, is homogeneous with respect to σ, i.e $R_a^i(g;r\sigma) = r^i R_a^i(g;\sigma)$ for all $r \in \mathbb{R}$. This leads to the following definition.

Definition 3.32 *Let $D(g)$ be a geometric differential operator with conformal weight w, and let $\hat{g} = e^{2\sigma} g$ conformally equivalent to g.*

$D(g)$ is called conformally covariant of bi-degree (a,b) if for all metrics g and $\sigma \in \mathcal{C}^\infty(M)$ it holds that $R_a(g;\sigma) = 0$.

$D(g)$ is called infinitesimally conformally covariant of bi-degree (a,b) if for all metrics g and $\sigma \in \mathcal{C}^\infty(M)$ it holds that $R_a^1(g;\sigma) = 0$.

Remark 3.33 One could generalize the definition of conformal covariance to geometric differential operators acting between arbitrary vector bundles. Note, if the involved vector bundles are metric dependent, one has to take into account the corresponding identification of these vector bundles, such as for spinor bundles the map F_σ it does.

A first example of a conformally covariant operator is given by the Dirac operator $D\!\!\!\!/ : \Gamma(S(M,g)) \to \Gamma(S(M,g))$ which has bi-degree $(\frac{1-n}{2}, -\frac{n+1}{2})$. Note that if a conformally covariant differential operator of bi-degree (a,b) has a well defined order, which will almost be the case in this thesis, the difference $a-b$ gives back the order of the operator.

Proposition 3.34 *Let $D(g)$ be a geometric differential operator. $D(g)$ is conformally* *covariant of bi-degree (a,b) if and only if $D(g)$ is infinitesimal conformally covariant of bi-degree (a,b).*

Proof. That fact that conformal covariance implies infinitesimal conformal covariance is clear from the definitions.

Now we prove the converse. Let $D(g)$ be infinitesimally conformally covariant of bi-degree (a,b). We have to show that $R_a(g,\sigma)=0$ for all metrics g and $\sigma\in\mathcal{C}^\infty(M)$. Now consider a $1-$parameter family of functions $\sigma_t\in\mathcal{C}^\infty(M)$ for $t\in[0,1]$, such that $\sigma_0=0$ and $\sigma_1=\sigma$, and the induced metrics $g_t:=e^{2\sigma_t}g$. To conclude $R_a(g,\sigma)=0$, it is enough to show that $e^{-b\sigma_t}F_{\sigma_t}\circ D(g_t)(e^{a\sigma_t}F_{\sigma_t})$ is constant along t, because in $t=0$ we get $D(g)$, hence $R_a(g;\sigma)=0$. Since g and σ were arbitrary, the result will follow. Now a computation shows

$$\begin{aligned}\frac{d}{dt}_{|t=t_0}&\big[e^{-b\sigma_t}F^{-1}_{\sigma_t}\circ D(g_t)(e^{a\sigma_t}F_{\sigma_t}\psi)\big]\\ &=\frac{d}{dh}_{|h=0}\big[e^{-b\sigma_{t_0+h}}F^{-1}_{\sigma_{t_0+h}}\circ D(g_{t_0+h})(e^{a\sigma_{t_0+h}}F_{\sigma_{t_0+h}}\psi)\big].\end{aligned}$$

For $g_{t_0+h}=e^{2(\sigma_{t_0+h}-\sigma_{t_0})}g_{t_0}$ the equation (3.16) reads

$$\begin{aligned}e^{-b(\sigma_{t_0+h}-\sigma_{t_0})}F^{-1}_{\sigma_{t_0+h}-\sigma_{t_0}}&\circ D(e^{2(\sigma_{t_0+h}-\sigma_{t_0})}g_{t_0})(e^{a(\sigma_{t_0+h}-\sigma_{t_0})}F_{\sigma_{t_0+h}-\sigma_{t_0}}\psi)\\ &=D(g_{t_0})+R_a(g_{t_0};\sigma_{t_0+h}-\sigma_{t_0})\psi.\end{aligned}$$

Due to $F_{\sigma_{t_0+h}-\sigma_{t_0}}=F_{\sigma_{t_0+h}}\circ F^{-1}_{\sigma_{t_0}}$ we may derive

$$\begin{aligned}\frac{d}{dt}_{|t=t_0}&\big[e^{-b\sigma_t}F^{-1}_{\sigma_t}\circ D(g_t)(e^{a\sigma_t}F_{\sigma_t}\psi)\big]\\ &=\frac{d}{dh}_{|h=0}\big[e^{-b\sigma_{t_0}}F^{-1}_{\sigma_{t_0}}\circ R_a(g_{t_0};\sigma_{t_0+h}-\sigma_{t_0})(e^{a\sigma_{t_0}}F_{\sigma_{t_0}}\psi)\big].\end{aligned}$$

Considering the splitting of $R_a(g_{t_0};\sigma_{t_0+h}-\sigma_{t_0})$ into homogenous parts leads to

$$\begin{aligned}\frac{d}{dh}_{|h=0}&\big[e^{-b\sigma_{t_0}}F^{-1}_{\sigma_{t_0}}\circ R_a(g_{t_0};\sigma_{t_0+h}-\sigma_{t_0})(e^{a\sigma_{t_0}}F_{\sigma_{t_0}}\psi)\big]\\ &=e^{-b\sigma_{t_0}}F^{-1}_{\sigma_{t_0}}\circ R^1_a(g_{t_0};\frac{d}{dh}_{|h=0}\sigma_{t_0+h})(e^{a\sigma_{t_0}}F_{\sigma_{t_0}}\psi).\end{aligned}$$

This expression vanishes by assumption and the proof is complete. □

Let us point out why the infinitesimal version has some advantages from the computational point of view. Let $\sigma\in\mathcal{C}^\infty(M)$ be arbitrarily fixed and consider the $1-$parameter family of metrics $g_t=e^{2t\sigma}g$ for $t\in[0,1]$. In case of g_t, equation (3.16) reads

$$F^{-1}_{t\sigma}(e^{-bt\sigma}D(g_t)(e^{at\sigma}F_{t\sigma}\psi))=D(g)\psi+R_a(g;t\sigma)\psi.$$

Taking the derivative with respect to t in $t = 0$ gives

$$\frac{d}{dt}_{|t=0}\left[F_{t\sigma}^{-1}(e^{-bt\sigma}D(g_t)(e^{at\sigma}F_{t\sigma}\psi))\right] = R_a^1(g,\sigma), \tag{3.18}$$

thus, this might be a way for computing $R_a^1(g,\sigma)$, hence checking the infinitessimal conformal covariance. This method will be used in Chapter 6.

4 Construction of conformally covariant differential operators via spectral theory

This chapter is devoted to a spectral theoretical construction of conformal powers of the Dirac operator. They will arise in the asymptotic expansion of a formal eigenspinor with respect to the Dirac operator of the Poincaré model [GMP10, GMP12]. This method of construction is inspired by the work of Graham and Zworski [GZ03]. Many parts of this chapter were obtained nearly parallel to [GMP10, GMP12]. Since computations are quite complicated and not given in a very explicit form in [GMP10, GMP12], we will recall their work, concerning the conformal powers of the Dirac operator, in detail.

To derive a suitable spin calculus for embedded submanifolds of codimension one, we first have to analyse the algebraic setup inside the Clifford calculus with respect to $\mathbb{R}^n$ sitting in $\mathbb{R}^{n+1}$ and to investigate the corresponding spin structures. This framework will be used to find a relationship between the corresponding Dirac operators. Then, the main point of this chapter is to solve formally the eigenequation on the Poincaré model. Finally, we indentify conformal powers of the Dirac operator inside the asymptotic expansion of those formal eigenspinor.

4.1 Spin calculus for hypersurfaces

Dealing with embedded submanifolds inside the spin calculus environment needs the specification of the corresponding representations of the spin groups, and the spin structures. One ends up with a hypersurface theory for spinor bundles, e.g. Gauss equation. For an embedded submanifold of codimension greater than one we refer to [Bär98], whereas details concerning codimension one can be found in [Bur93] or [Bau81], but we will mainly follow [BGM05].

Let us start to identify the representations. The inclusion $\mathbb{R}^{p,q} \subset \mathbb{R}^{p,q+1}$, where $\mathbb{R}^{p,q+1} = span\{e_1, \ldots, e_n, e_{n+1}\}$, leads to an algebra isomorphism $\Phi : \mathcal{C}_{p,q} \simeq \mathcal{C}^0_{p,q+1}$ induced by $u(x) := e_{n+1} \cdot x$. The Clifford algebra $\mathcal{C}_{p,q+1}$ carries an unique (up to equivalence) irreducible representation $\Xi_{p,q+1} : \mathcal{C}_{p,q+1} \to End(\mathbb{C}^{2^m})$ in case of $p + q + 1 = 2m$ is even, whereas in case $p + q + 1 = 2m + 1$ there are exactly (up to equivalence) two non-equivalent irreducible representations $\Xi^0_{p,q+1}, \Xi^1_{p,q+1} : \mathcal{C}_{p,q+1} \to End(\mathbb{C}^{2^m})$. Compositions with Φ yield irreducible representations $\Xi_{p,q+1} \circ \Phi$ and $\Xi^i_{p,q+1} \circ \Phi$, $i = 0, 1$, of $\mathcal{C}_{p,q}$ into C^{2^m}. By definition $\Xi_{p,q+1} \circ \Phi = \bar{\Xi}^0_{p,q} + \bar{\Xi}^1_{p,q}$ decomposes into two non equivalent irreducible representations on $\mathbb{C}^{2^{m-1}}$, whereas $\Xi^i_{p,q+1} \circ \Phi$, $i = 0, 1$, are irreducible and equivalent due to the fact that $\Xi^i_{p,q+1}$, $i = 0, 1$, become equivalent when restricted to the even subalgebra $\mathcal{C}^0_{p,q+1}$. Because of uniqueness, we may conclude that the $\bar{\Xi}^i_{p,q}$ are equivalent to $\Xi^i_{p,q}$, $i = 0, 1$, and $\Xi_{p,q+1} \circ \Phi$ is equivalent to $\Xi_{p,q}$. Let us denote by $\bar{\kappa}_{p,q} : Spin_0(p,q) \to Gl(\mathbb{C}^{2^m})$ the restriction of the irreducible representation $\bar{\Xi}^0_{p,q+1}$, for $p+q+1 = 2m$, or $\Xi^0_{p,q+1} \circ \Phi$, for $p+q+1 = 2m+1$, of $\mathcal{C}_{p,q}$ to the spin group $Spin_0(p,q)$. Again, in case of $p+q+1 = 2m$ even, it does not matter which irreducible representation we have taken, because they become equivalent when resticted to the even subalgebra $\mathcal{C}^0_{p,q}$. We have the following two cases: If $p+q+1 = 2m$ is even, $\bar{\kappa}_{p,q} = \bar{\kappa}^0_{p,q} \oplus \bar{\kappa}^1_{p,q}$ decomposes into two irreducible representations $\bar{\kappa}^i_{p,q} : Spin_0(p,q) \to Gl(\mathbb{C}^{2^{m-1}})$, $i = 0, 1$, which are equivalent, hence equivalent to $\kappa_{p,q}$. If $p+q+1 = 2m+1$ is odd, $\bar{\kappa}_{p,q} = \bar{\kappa}^+_{p,q} \oplus \bar{\kappa}^-_{p,q}$ decomposes into two non-equivalent irreducible representations $\bar{\kappa}^+_{p,q}, \bar{\kappa}^-_{p,q} : Spin_0(p,q) \to Gl(\mathbb{C}^{2^{m-1}})$, hence equivalent to $\kappa^{\pm}_{p,q}$ (note that $p+q = 2m$).

Let $(N^{p,q+1}, h)$ be an semi Riemannian spin manifold. Consider an embedded submanifold $(M, \iota : M \to N, g := \iota^* h, \nu)$ of N with the induced semi Riemannian metric, induced time- and space orientation, and spacelike unit global normal vector field ν. Any spin structure $(\mathcal{Q}^N, f^N)$ of (N, h) induces a spin structure on (M, g) as follows: Considering the inclusion $SO_0(p,q) \subset SO_0(p,q+1)$ yields an embedding $i : \mathcal{P}^g \to \mathcal{P}^h_{|M}$ defined by $i_x(s_1, \ldots, s_n) := (d\iota_x(s_1), \ldots, d\iota_x(s_n), \nu)_{\iota(x)}$ which commutes with the group actions. Then

$$\left(\mathcal{Q}^M := (f^N)^{-1}[i(\mathcal{P}^g)], f^M := f^N_{|\mathcal{Q}^M}\right)$$

is a spin structure on (M, g). Now let us consider the spinor bundles corresponding to the given structures above. Let us assume that (N, h) is spin and consider on (M, g) the induced spin structure. The restriction of $S(N, h) := \mathcal{Q}^N \times_{Spin_0(p,q+1)} \Delta_{p,q+1}$ to M yields an isomorphism

$$S(N,h)_{|M} = (\mathcal{Q}^N \times_{Spin_0(p,q+1)} \Delta_{p,q+1})_{|M} \simeq \mathcal{Q}^M \times_{Spin_0(p,q)} \Delta_{p,q+1},$$

since $\mathcal{Q}^M$ is an $Spin_0(p,q) \subset Spin_0(p,q+1)-$reduction of $\mathcal{Q}^N_{|M}$. From the algebraic considerations done before we obtain

$$S(N,h)_{|M} = \begin{cases} S(M,g), & n = 2m \\ S(M,g) \oplus S(M,g), & n = 2m+1. \end{cases} \tag{4.1}$$

Note that in case $n = 2m+1$, the spinor bundles are the same but the Clifford multiplications differ by a sign, linked to the non-equivalence of $\bar{\Xi}^0_{p,q}$ and $\bar{\Xi}^1_{p,q}$ on $\mathcal{C}_{p,q}$.

Some results from the theory of hypersurfaces in semi Riemannian geometry can be carried over to spin geometry. First let us fix some notation: The vector fields of N along the embedding ι are denoted by

$$\mathfrak{X}_\iota(N) := \{X \in \mathcal{C}^\infty(M, TN) \mid X(x) \in T_{\iota(x)}N \quad \forall x \in M\}.$$

Denoting the tangential and normal vector fields along the embedding ι by $\mathfrak{X}_\iota(N)^{tan}$

and $\mathfrak{X}_\iota(N)^{nor}$, respectively, we get projection maps

$$tan : \mathfrak{X}_\iota(N) \to \mathfrak{X}_\iota(N)^{tan},$$
$$nor : \mathfrak{X}_\iota(N) \to \mathfrak{X}_\iota(N)^{nor}.$$

We will identify $\mathfrak{X}(M)$ with $\mathfrak{X}_\iota(N)^{tan}$ by

$$\mathfrak{X}(M) \ni X \mapsto d\iota_x\left(X(x)\right) \in \mathfrak{X}_\iota(N)^{tan}.$$

For $X \in \mathfrak{X}(M)$ the tangential and normal projection of the Levi-Civita connection ∇^N, i.e., $tan(\nabla^N_X) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ and $nor(\nabla^N_X) : \mathfrak{X}_\iota(N)^{nor} \to \mathfrak{X}_\iota(N)^{nor}$, yield the Levi-Civita connection ∇^M and a covariant derivative $\nabla^\perp$ on the normal bundle $(NM, \pi, M, \mathbb{R})$. Note that in our case the normal bundle is parallelizable through the normal field ν. The second fundamental form on M is given by

$$II_x(X,Y) := h_x(\nabla^N_X Y, \nu)$$

for $X, Y \in \mathfrak{X}(M)$. The equation of Gauss is then

$$(\nabla^N_X Y)_x = (\nabla^M_X Y)_x + II_x(X,Y)\nu_x, \tag{4.2}$$

where $X, Y \in \mathfrak{X}(M)$. The next ingredient we need is the Weingarten map

$$A_\nu : \mathfrak{X}(M) \to \mathfrak{X}(M),$$

defined by the equation $g_x(A_\nu X, Y) = II_x(X,Y)$, for all $X, Y \in \mathfrak{X}(M)$. Since the second fundamental form II is symmetric, it follows that A_ν is formally self-adjoint with respect to g. Thus, the Weingarten equation is

$$(\nabla^N_X \eta)_{\iota(x)} = (\nabla^\perp_X \nu)_x - (A_\nu X)_x \tag{4.3}$$

for $X \in \mathfrak{X}(M)$ and $x \in M$. Based on local computations using the equation of Gauss (4.2) we obtain:

Proposition 4.1 *Let $\phi \in \Gamma\left(S(N^{n+1}, h)\right)$ and $X \in \mathfrak{X}(M)$. Then one has*

$$(\nabla^{S(N,h)}_X \phi)_{|M} = \tilde{\nabla}^{S(M,g)}_X \phi - \frac{1}{2}\nu \cdot A_\nu(X) \cdot \phi, \tag{4.4}$$

where $\tilde{\nabla}^{S(M,g)}$ denotes $\nabla^{S(M,g)}$ in case $n = 2m$ even, whereas for $n = 2m+1$ odd it is $\begin{pmatrix} \nabla^{S(M,g)} & 0 \\ 0 & \nabla^{S(M,g)} \end{pmatrix}$. In particular, one has the following equality for any $\phi \in \Gamma\left(S(N,h)\right)$,

$$(\not{D}^N \phi)_{|M} = -\nu \cdot \tilde{\not{D}}^g \phi - \frac{1}{2} tr(A_\nu)\nu \cdot \phi + \nu \cdot \nabla^{S(N,h)}_\nu \phi, \tag{4.5}$$

where $\tilde{\not{D}}^g \phi \overset{loc.}{=} \sum_i \varepsilon_i \nu \cdot s_i \cdot \tilde{\nabla}^{S(M,g)}_{s_i} \phi$.

Just as $\tilde{\nabla}^{S(M,g)}$, the operator $\tilde{\not{D}}^g$ considered with respect to the identification (4.1) is given by the Dirac operator $\not{D}^M$ of (M,g) in case $n = 2m$ even, and $\begin{pmatrix} \not{D}^M & 0 \\ 0 & -\not{D}^M \end{pmatrix}$ in case $n = 2m+1$ odd. For simplification note that the trace of the Weingarten operator is up to a factor the mean curvature $H := \frac{1}{n} tr_g(A_\nu)$. Another important formula is given by:

Proposition 4.2 *Let $\phi \in \Gamma(S(N,h))$. Then one has*

$$\left([\nabla^{S(N,h)}_\nu, \tilde{\not{D}}^g]\phi\right)_{|M} = \mathcal{D}^{A_\nu}\phi - \frac{n}{2}\nu \cdot \mathrm{grad}^g(H) \cdot \phi + \frac{1}{2}\nu \cdot div^M(A_\nu) \cdot \phi, \tag{4.6}$$

where $\mathcal{D}^{A_\nu} := \sum_{i=1}^n \varepsilon_i \nu \cdot s_i \cdot \tilde{\nabla}^{S(M,g)}_{A_\nu(s_i)}$ *and* $div^M(A_\nu) := \sum_{i=1}^n \varepsilon_i (\nabla^M_{s_i} A_\nu)(s_i)$ *with respect to a local section* $(s_1, \ldots, s_n) : U \to \mathcal{P}^g$.

The proof is based on computations using many formulas from hypersurface theory, like Codazzi, Gauss and Ricatti mixed with the standard formulas for spin geometry, see [BGM05, Proposition 3.1].

The formula from equation (4.6) will be used in the computation of the first variation of the Dirac operator with respect to a 1−parameter family of metrics g_t. Thus, let M^n be a manifold with a 1−parameter family of semi Riemannian metrics g_t, $t \in \mathbb{R}$, and consider its generalized cylinder $(Z^{n+1} := M \times \mathbb{R}, h := dt^2 + g_t)$. For all $t \in \mathbb{R}$ let us denote by $(M_t := M \times \{t\}, \iota_t : M_t \to Z, g_t := \iota_t^* h, \nu_t)$ the embedded submanifold of N with induced semi Riemannian metric and orientation, and spacelike unit global normal vector field $\nu_t := \partial_t \in \mathfrak{X}^{nor}_{\iota_t}(Z)$. The parallel transport

$$\tau_t^{t_0} := \mathcal{P}^{S(Z,h),\tilde{A}^h}_{t,t_0} : S(Z,h)_{(t,x)} \to S(Z,h)_{(t_0,x)}$$

along the curve $\gamma_x(t) = (t,x) \in Z$ for a fixed $x \in M$ makes it possible to relate spinors on (M, g_t) for different t's. It respects Clifford multiplication with vector fields tangential to M_t, i.e, for $\phi \in \Gamma(S(Z,h)_{|M_t})$ and $X \in \mathfrak{X}_{\iota_t}(Z)^{tan}$, it holds that

$$\tau_t^{t_0}(X \cdot_t \phi) = \tau_t^{t_0}(X) \cdot_{t_0} \tau_t^{t_0}\phi,$$

where we denote the parallel transport of X inside $\mathfrak{X}(Z)$ along the curve γ_x by the same symbol. The formula (4.6) contains the variation of the Dirac operator with respect to g_t. Namely, for $\phi \in \Gamma(S(Z,h))$ it follows that

$$\left([\nabla^{S(Z,h)}_{\nu_t}, \tilde{\not{D}}^{g_t}]\phi\right)_{|M_t} = \nabla^{S(Z,h)}_{\nu_t}(\tilde{\not{D}}^{g_t}\phi) = \frac{d}{ds}_{|s=t}(\tau_s^t \circ \tilde{\not{D}}^{g_s} \circ \tau_t^s \phi), \tag{4.7}$$

which yields for $t = 0$ the first variation of the Dirac operator with respect to g_t.

Remark 4.3 In [BG92] the authors obtained in a different way the same result for the first variation of the Dirac operator with respect to $g_t = g + tk$, where g is a Riemannian metric and k is a symmetric bilinear map.

Remark 4.4 One can define a Taylor-like series for Dirac operators with respect to a $1-$parameter family of semi Riemannian metrics g_t on a manifold M: In the setting of generalized cylinders we have worked within the spinor bundle $S(Z,h)$. From equation (4.7) we can consider

$$\tilde{D\!\!\!/}^{g_t} = \tau_0^t \circ \tilde{D\!\!\!/}^{g_0} \circ \tau_t^0 + t \frac{d}{ds}_{|s=t} (\tau_s^t \circ \tilde{D\!\!\!/}^{g_s} \circ \tau_t^s) + O(t^2).$$

With respect to the splitting (4.1) this leads to a formal power series of the Dirac operator on (M_t, g_t) by

$$D\!\!\!/^{g_t} = \tau_0^t \circ D\!\!\!/^{g_0} \circ \tau_t^0 + \sum_{i=1}^{\infty} \frac{t^i}{i!} \frac{d^i}{ds^i}_{|s=t} (\tau_s^t \circ D\!\!\!/^{g_s} \circ \tau_t^s),$$

for $n = 2m$ even, and for $n = 2m+1$ we derive twice that formula with reversed sign. This will be considered as a Taylor-like series for Dirac operators with respect to a g_t, which will play an important role later in this section.

Let us present an important example.

Example 4.5 Consider the Poincaré model (X^+, g_+) of a conformal manifold (M, c), see Remark 3.29. Fixing a metric $h \in c$ induces a radial coordinate r, hence a unit normal vector field $r\partial_r$, such that $h_r = h - r^2 P^h + r^4 h^{(4)} + \ldots$ is formally a $1-$parameter family of metrics on M. The compactification $(\bar{X}^+, \bar{g} := r^2 g_+)$ carries the structure of a generalized cylinder. In this particular metric the Weingarten map A_{∂_r} with respect to (M,h) is just $-P^h$, and therefore $H = \frac{1}{n} J^h$ is the mean curvature. Since h_r is even in the power of r, the first variation of the Dirac operator with respect to h_r vanishes, whereas its second variation is given by the formula

$$\frac{d^2}{dr^2}_{|r=0} (\tau_r^0 \circ D\!\!\!/^{h_r} \circ \tau_0^r \phi) = \sum_{i=1}^{n} \varepsilon_i P^h(s_i)^\natural \cdot \nabla_{s_i}^{S(M,h)} \phi = (P^h, \nabla^{S(M,h)} \phi). \qquad (4.8)$$

Here we made use of $\delta^{\nabla^{LC}} P^h = -dJ^h$, hence the second and the third summand in (4.6) cancel each other.

4.2 The Dirac operator of the Poincaré model

Let (M, c) be a conformal spin manifold. Denote by $(X^+ = (0, \varepsilon) \times M, g_+, r)$ the Poincaré model of (M, h) for $h \in c$. This means $g_+ = r^{-2}(dr^2 + h_r)$ is a Einstein metric on the interior of X^+, where $r : X^+ \to \mathbb{R}^+$ is the defining function for the boundary $\{0\} \times M$

induced by h, and h_r is a 1−parameter family of metrics on M with $h_0 = h$.

The goal of the next section is to extend formally a spinor on M to a spinor on X^+ solving formally the eigenspinor equation for the Dirac operator on X^+. Since g_+ becomes singular at $r = 0$, it make no sense of restricting the spinor bundle of X^+ to M. A way to overcome that problem is to work within the compactification

$$\left(\bar{X}^+ := [0, \varepsilon) \times M, \bar{g} := r^2 g_+ = dr^2 + h_r\right).$$

Since g_+ and $\bar{g}$ are conformally equivalent on the interior of $\bar{X}^+$ we have the following identification:

$$F_r := F_{\log(r)} : S(X^+, g_+) \to S(X^+, \bar{g}). \tag{4.9}$$

To keep an overview of these different bundles, we will write θ for spinors on $(\bar{X}^+, \bar{g})$, φ for spinors on (X^+, g_+) and ψ for spinors on the boundary of $(\bar{X}^+, \bar{g})$ as well as for spinors on (M, h), respectively. Let $\psi \in \Gamma(S(\bar{X}^+, \bar{g})_{|M})$ be a spinor field. A spinor field $\varphi \in \Gamma\big(S(X^+, g_+)\big)$ is called an extension of ψ, if there exists a spinor field $\theta \in \Gamma\big(S(\bar{X}^+, \bar{g})\big)$, such that $\theta(0, x) = \psi(0, x)$ and $\theta(r, x) = F_r\big(\varphi(r, x)\big)$ for $x \in M$ and $r \in (0, \varepsilon)$. We will get formal existence and uniqueness results for the extension procedure by demanding that it formally solves the eigenspinor equation $\not{D}^{X^+}\varphi = i\lambda\varphi$.

Adapting equation (4.5) to our situation, we obtain

$$\left(\not{D}^{\bar{X}^+}\theta\right)_{|M_r} = -\nu \cdot \tilde{\not{D}}^{h_r}\psi - \frac{n}{2}H_r\nu \cdot \psi + \nu \cdot \nabla_\nu^{S(\bar{X}^+, \bar{g})}\psi,$$

where $\theta \in \Gamma\left(S(\bar{X}^+, \bar{g})\right)$, $M_r = \{r\} \times M$ denotes the level r−leaf with metric h_r, and H_r denotes the mean curvature of M_r with respect to $\nu := \partial_r$. The conformal covariance property of $\not{D}^{\bar{X}^+}$ restricted to the interior of $\bar{X}^+$ yields a relation to the Dirac operator of the Poincaré model.

Lemma 4.6 *The Dirac operator on (X^+, g_+) is given by*

$$\begin{aligned}\not{D}^{X^+}\varphi = &- r(r\nu) \cdot F_r^{-1}\big(\tilde{\not{D}}^{h_r}(F_r\varphi)\big) - r\frac{n}{2}H_r(r\nu) \cdot \varphi \\ &+ r(r\nu) \cdot F_r^{-1}\big(\nabla_\nu^{S(\bar{X}^+, \bar{g})}(F_r\varphi)\big) - \frac{n}{2}(r\nu) \cdot \varphi,\end{aligned} \tag{4.10}$$

where $\varphi \in \Gamma\big(S(X^+, g_+)\big)$.

Proof. The evaluation of the conformal covariance of the Dirac operator and $F_r(X\cdot) = r^{-1}X\hat{\cdot}$ gives

$$\not{D}^{X^+}\varphi = r^{\frac{n+2}{2}}F_r^{-1}\big(\not{D}^{\bar{X}^+}(r^{-\frac{n}{2}}F_r\varphi)\big)$$

$$
\begin{aligned}
&= r^{\frac{n+2}{2}} F_r^{-1}\big(-\nu \cdot \tilde{\not{D}}^{h_r}(r^{-\frac{n}{2}} F_r\varphi) - \frac{n}{2} r^{-\frac{n}{2}} H_r \nu \cdot F_r\varphi + \nu \cdot \nabla_\nu^{S(\bar{X}^+,\bar{g})}(e^{-\frac{n}{2}} F_r\varphi)\big) \\
&= - r^2 \nu \cdot F_r^{-1}(\tilde{\not{D}}^{h_r}(F_r\varphi)) - \frac{n}{2} r^2 H_r \nu \cdot \varphi - \frac{n}{2} r\nu \cdot \varphi + r^2 \nu \cdot F_r^{-1}(\nabla_\nu^{S(\bar{X}^+,\bar{g})} F_r\varphi).
\end{aligned}
$$

□

We have thus calculated how the Dirac operator of the Poincaré model looks like. To solve an eigenspinor equation of the Dirac operator $\not{D}^{X^+}$, we will introduce a splitting of the spinor bundle $S(\bar{X}^+, \bar{g})$ which is independent of the dimension n. To do this, note that $\nu^2 = -1$, hence $S(\bar{X}^+, \bar{g})$ decomposes into the $\nu-$eigenspaces

$$
S^{\pm\partial_r}(\bar{X}^+, \bar{g}) := \{\theta \in S(\bar{X}^+, \bar{g}) \mid \nu \cdot \theta = \pm i\theta\}, \tag{4.11}
$$

hence giving

$$
S(\bar{X}^+, \bar{g}) \simeq S^{+\partial_r}(\bar{X}^+, \bar{g}) \oplus S^{-\partial_r}(\bar{X}^+, \bar{g}).
$$

Let us denoted by $S^{\pm\partial_r}(X^+, \bar{g})$ the restriction of the $\nu-$eigenspaces $S^{\pm\partial_r}(\bar{X}^+, \bar{g})$ to the interior of $\bar{X}^+$. In the same way we may consider the splitting of $S(X^+, g_+)$ with respect to unit normal vector field $r\partial_r$, i.e.,

$$
S(X^+, g_+) \simeq S^{+r\partial_r}(X^+, g_+) \oplus S^{-r\partial_r}(X^+, g_+),
$$

where $S^{\pm r\partial_r}(X^+, g_+) := \{\varphi \in S(X^+, g_+) \mid r\partial_r \cdot \varphi = \pm i\varphi\}$. These splittings obey the following properties:

Lemma 4.7 *The spinor bundle isomorphism F_r respects the splittings, i.e., for any $\varphi^\pm \in \Gamma(S^{\pm r\partial_r}(X^+, g_+))$ it holds that $F_r(\varphi^\pm) \in \Gamma(S^{\pm\partial_r}(X^+, \bar{g}))$.*

Furthermore, for $\varphi^\pm \in \Gamma\left(S^{\pm r\partial_r}(X^+, g_+)\right)$, $\theta^\pm \in \Gamma\left(S^{\pm\partial_r}(\bar{X}^+, \bar{g})_{|M_r}\right)$ and $X \in \mathfrak{X}(M_r)$ one has

(1) $X \cdot \varphi^\pm \in \Gamma\left(S^{\mp r\partial_r}(X^+, g_+)\right)$ and $X \cdot \theta^\pm \in \Gamma\left(S^{\mp\partial_r}(\bar{X}^+, \bar{g})_{|M_r}\right)$.

(2) $\tilde{\nabla}_X^{S(M,h_r)}\theta^\pm \in \Gamma\left(S^{\pm\partial_r}(\bar{X}^+, \bar{g})_{|M_r}\right)$.

In particular, the operator $\tilde{\not{D}}^{h_r}$ interchanges the spaces $\Gamma(S^{\pm\partial_r}(X^+, g))$.

Proof. First we compute

$$
\nu \cdot F_r(\varphi^\pm) = rr^{-1}\partial_r \cdot F_r(\varphi) = rF_r(\partial_r \cdot \varphi^\pm) = F_r(r\partial_r \cdot \varphi^\pm) = \pm iF_r(\varphi^\pm).
$$

This gives $F_r(\varphi^\pm) \in \Gamma(S^{\pm\partial_r}(X^+, g))$. The claim in (1) is obvious, since a vector field tangential to M_r anticommutes with the normal vector field ν. The second statement follows from the equations (4.3) and (4.4), since for any $\theta^\pm \in \Gamma\left(S^{\pm\partial_r}(\bar{X}^+, \bar{g})_{|M_r}\right)$ we have that

$$\begin{aligned}\nu\cdot\tilde{\nabla}^{S(M,h_r)}_X\theta^{\pm} =&\nu\cdot(\nabla^{S(\bar{X}^+,\bar{g})}_X\theta^{\pm})_{|M_r}+\frac{1}{2}\nu\cdot\nu\cdot A_\nu(X)\cdot\theta^{\pm}\\ =&\Big(\nabla^{S(\bar{X}^+,\bar{g})}_X(\nu\cdot\theta^{\pm})\Big)_{|M_r}-\nabla^{\bar{g}}_X\nu\cdot\theta^{\pm}-\frac{1}{2}\nu\cdot A_\nu(X)\cdot\nu\cdot\theta^{\pm}\\ =&\Big(\nabla^{S(\bar{X}^+,\bar{g})}_X(\nu\cdot\theta^{\pm})\Big)_{|M_r}+\frac{1}{2}\nu\cdot A_\nu(X)\cdot\nu\cdot\theta^{\pm}\\ =&\pm i\tilde{\nabla}^{S(M,h_r)}_X\theta^{\pm}.\end{aligned}$$

The last statement about the Dirac operator is a consequence of the last two statements. □

Now we have all the tools needed to formally solve the eigenspinor equation for the Dirac operator $\not{D}^{X^+}$. This will be done in the next Section.

4.3 The eigenspinor problem on the Poincaré model

In the setting of Section 4.2 let us consider the eigenspinor equation of the Dirac operator on (X^+,g_+) given by

$$\not{D}^{X^+}\varphi=\lambda\varphi \tag{4.12}$$

for $\varphi\in\Gamma\big(S(X^+,g_+)\big)$ and $\lambda\in\mathbb{C}$. This is equivalent under the identification F_r, equation (4.9), to

$$F_r(\not{D}^{X^+}\varphi)=\lambda F_r(\varphi).$$

Furthermore, by Lemma 4.6, this is equivalent to the restriction of

$$D(\bar{g})\theta=\lambda\theta \tag{4.13}$$

to the interior of $\bar{X}^+$, where we have defined

$$D(\bar{g})\theta:=-r\nu\cdot\tilde{\not{D}}^{h_r}\theta-\frac{n}{2}rH_r\nu\cdot\theta+r\nu\cdot\nabla^{S(\bar{X}^+,g)}_\nu\theta-\frac{n}{2}\nu\cdot\theta$$

for $\theta:=F_r\varphi\in\Gamma\big(S(X^+,\bar{g})\big)$. Hence, a solution θ of equation (4.13) gives us a solution $\varphi:=F_r^{-1}\theta$ of equation (4.12).

In what follows, we are only interested in formal solutions of (4.13) in a neighbourhood of the boundary $\partial\bar{X}^+=M$. Hence we look for solutions from the space

$$\mathcal{A}:=\left\{\theta=\sum_{j=0}^{\infty}r^j\theta_j\,\middle|\,\theta_j\in\Gamma\big(S(\bar{X}^+,\bar{g})_{|M\times[0,\varepsilon)}\big):\quad\nabla^{S(\bar{X}^+,\bar{g})}_\nu\theta_j=0\right\}.$$

Thus we can always assume, that for $0\le j<\infty$ the section θ_j arises from a section of the boundary M by parallel transport along the geodesic $\gamma_x(r) := (x, r)$. The structure of the operator $D(\bar{g})$, in view of Lemma 4.7, and the formal power series for H_r and $\tilde{\not{D}}^{h_r}$ with respect to r, which are given by

$$H_r = \sum_{k=0}^{\infty} \frac{1}{k!} r^k H_r^k, \quad \tilde{\not{D}}^{h_r} = \sum_{k=0}^{\infty} \frac{1}{k!} r^k \tilde{\not{D}}^{h,k}, \tag{4.14}$$

leads to a decomposition of $\mathcal{A}$ into

$$\mathcal{A}^{\pm} := \{\theta \in \mathcal{A} \mid \theta_{2j} \in \Gamma(S^{\pm\partial_r}(\bar{X}^+, \bar{g})),\ \theta_{2j+1} \in \Gamma(S^{\mp\partial_r}(\bar{X}^+, \bar{g}))\},$$

which is invariant under the action of $D(\bar{g})$, i.e., $D(\bar{g})\mathcal{A}^{\pm} \subset \mathcal{A}^{\pm}$, due to the fact that H_r is an odd formal power series in r, whereas $\tilde{\not{D}}^{h_r}$ is an even formal power series in r. Here we denoted by $\tilde{\not{D}}^{h,k}$ the $k-$th variation of the Dirac operator with respect to h_r. Note, that an explicit knowledge of each summand in the above formal power series for H_r and $\tilde{\not{D}}^{h_r}$ is an illusive task, as the explicit form of h_r and higher variations of the Dirac operator with respect to h_r are unknown. But, we already have calculated the first two variations of the Dirac operator with respect to h_r, see Example 4.5. Now, we compute the first summands of the formal power series of $H_r = \frac{1}{n} tr_{h_r} A_\nu$, $\nu = \partial_r$. First one observes:

Lemma 4.8 *The Weingarten map A_ν on M_r is formally given by*

$$(h_r)_x(A_\nu(X), Y) = -\frac{1}{2} \frac{d}{dt}_{|t=r} (h_t)_x(X, Y),$$

where $X, Y \in T_{(r,x)} M_r$ and $(r, x) \in M_r$.

Proof. Let $(r, x) \in M_r$ and $X, Y \in T_{(r,x)} M_r$. We extent X and Y to vector fields in a neighbourhood of (r, x) in M_r such that their commutator vanishes at the point (r, x). Using $\bar{g}(X, \nu) = 0$ and $[X, \nu] = 0$ for vector fields $X \in \mathfrak{X}(M_r)$ we may derive from Koszul's formula that

$$\begin{aligned}(h_r)_x(A_\nu(X), Y) =& II_\nu(X, Y) = -\bar{g}_{(r,x)}(\nabla^{\bar{g}}_X \nu, Y) = -\frac{1}{2}\nu\left(\bar{g}_{(r,x)}(X, Y)\right) \\ =& -\frac{1}{2} \frac{d}{dt}_{|t=r} (h_t)_x(X, Y),\end{aligned}$$

which completes the proof. $\square$

From the knowledge of the first summands in the formal power series of $h_r = h - r^2 P + r^4 \frac{1}{4} h^{(4)} + O(r^6)$, we may derive the following:

Corollary 4.9 *The mean curvature H_r is formally given by*

$$H_r = \frac{1}{n} r \sum_i \varepsilon_i P^h(s_i^r, s_i^r) - \frac{2}{n} r^3 \sum_i \varepsilon_i h^{(4)}(s_i^r, s_i^r) + O(r^3),$$

where $\{s_i^r\}$ denotes an h_r-orthonormal basis.

In what will follow, we will identify M_0 with M.

Proposition 4.10 *Let $\lambda \notin -\mathbb{N} + \frac{1}{2}$ and $\psi^+ \in \Gamma(S^{+\partial_r}(\bar{X}^+, \bar{g})_{|M})$. Then there exists a unique $\theta \in \mathcal{A}^+$ such that $r^{\frac{n}{2}+\lambda}\theta$ is a solution of equation* (4.13) *with eigenvalue $i\lambda$ and $\theta_{|r=0} = \psi^+$.*

Proof. The proof is constructive: Consider an element $\theta \in \mathcal{A}$ and its splitting

$$\theta = \sum_j r^j \theta_j = \sum_j r^j (\theta_j^+ + \theta_j^-)$$

with respect to $S^{\pm\partial_r}(\bar{X}^+, \bar{g})$. Because of the nature of $D(\bar{g})$ we have to consider $\bar{\theta} := r^\mu \theta$ for a real number μ as candidate for the solution of equation (4.13) with eigenvalue $i\lambda$. Using all the formal expressions, we see that $D(\bar{g})\bar{\theta} = i\lambda\bar{\theta}$ is equivalent to

$$\begin{aligned}
i\lambda \sum_j r^{\mu+j}\theta_j =& - r\nu \cdot \sum_j r^{\mu+j} \tilde{\not{D}}^{h_r} \theta_j - \frac{n}{2} r \sum_j r^{\mu+j} H_r \nu \cdot \theta_j \\
&+ r\nu \cdot \sum_j \nabla_\nu (r^{\mu+j}\theta_j) - \frac{n}{2}\nu \cdot \sum_j r^{\mu+j}\theta_j \\
=& - \nu \cdot \sum_{k,j} r^{\mu+j+k+1} \tilde{\not{D}}^{h,k} \theta_j - \frac{n}{2}\nu \cdot \sum_{j,k} r^{\mu+j+k+1} H_r^k \cdot \theta_j \\
&+ \nu \cdot \sum_j (\mu+j) r^{\mu+j}\theta_j - \frac{n}{2}\nu \cdot \sum_j r^{\mu+j}\theta_j.
\end{aligned}$$

We will prove by induction that the coefficients θ_j are completely determined since we have an initial datum ψ^+. The basis is given by the $(\mu+0)-$th power of r in the equation given above:

$$i\lambda\theta_0 - \mu\nu \cdot \theta_0 + \frac{n}{2}\nu \cdot \theta_0 = 0.$$

Now $\theta_0 = \theta_0^+ + \theta_0^-$ can be determined by chosing $\mu = \frac{n}{2} + \lambda$, which sets up a freedom in θ_0^+ and implies $\theta_0^- = 0$. We set $(\theta_0^+)_{|M} := \psi^+$ and extend it to the interior by parallel transport along the $r-$geodesic to a spinor $\theta_0^+ \in \Gamma(S^{+\partial_r}(\bar{X}^+, \bar{g}))$. The induction step $n \to (n+1)$ is satisfied as long as the equation

$$i\lambda\theta_{n+1} = (\mu + n + 1)\nu \cdot \theta_{n+1} - \frac{n}{2}\nu \cdot \theta_{n+1} + P(\theta_{\leq n})$$

determines θ_{n+1} completely. Here $P(\theta_{\leq n}) = P^+(\theta_{\leq n}) + P^-(\theta_{\leq n})$ is a spinor which only depends on θ_k for $k \leq n$, it collects all the summands involving $\tilde{\not{D}}^{h,j}$ and H^j. For future reference, let us be more concrete here:

$$P(\theta_{\leq n}) = -\nu \cdot \sum_{k=0}^{n} \frac{1}{k!} \tilde{\not{D}}^{h,k} \theta_{n-k} - \frac{n}{2} \nu \cdot \sum_{k=0}^{n} \frac{1}{k!} H_r^k \theta_{n-k}. \tag{4.15}$$

Now we need the facts that $\tilde{\not{D}}^{h,k}$ interchange $\Gamma(S^{\pm\partial_r}(\bar{X}^+, \bar{g}))$ and is even in r as well as H_r is odd in r. The fact that θ_0 has no negative part with respect to ∂_r implies that in case $n = 2k$, one has that $P(\theta_{\leq n})$ has no positive part, i.e.,

$$P(\theta_{\leq n}) = 0 + P^-(\theta_{\leq n}),$$

whereas in case $n = 2k + 1$, one has that $P(\theta_{\leq n})$ has no negative part, i.e.,

$$P(\theta_{\leq n}) = P^+(\theta_{\leq n}) + 0,$$

thus only positive or negative parts do arise. The induction step splits under the action of ν into

$$\begin{aligned} \lambda\theta^+_{n+1} - (\mu + n + 1)\theta^+_{n+1} + \frac{n}{2}\theta^+_{n+1} - \frac{1}{i}P^+(\theta_{\leq n}) = 0, \\ \lambda\theta^-_{n+1} + (\mu + n + 1)\theta^-_{n+1} - \frac{n}{2}\theta^-_{n+1} - \frac{1}{i}P^-(\theta_{\leq n}) = 0. \end{aligned}$$

This implies in case $n = 2k$ that $\theta^+_{n+1} = 0$ and $(2\lambda + n + 1)\theta^-_{n+1} = \frac{1}{i}P^-(\theta_{\leq n})$, and for $n = 2k + 1$ we have $-(n + 1)\theta^+_{n+1} = \frac{1}{i}P^+(\theta_{\leq n})$ and $\theta^-_{n+1} = 0$. This determines θ_{n+1} in any case since $\lambda \notin -\mathbb{N} + \frac{1}{2}$.

By construction we found an unique $\theta \in \mathcal{A}^+$ such that $r^{\frac{n}{2}+\lambda}\theta$ solves equation (4.13) with eigenvalue $i\lambda$ and $\theta_{|r=0} = \psi^+$. □

Remark 4.11 If we change the assumption of Proposition 4.10 by starting with some $\psi^- \in \Gamma(S^{-\partial_r}(\bar{X}^+, \bar{g})_{|M})$, we end up with an unique $\theta \in \mathcal{A}^-$ such that $r^{\frac{n}{2}+\lambda}\theta$ solves equation (4.13) with eigenvalue $-i\lambda$ and $\theta_{|r=0} = \psi^-$.

We had to exclude some eigenvalues λ in Proposition 4.10. Dealing with them requires the consideration of logarithmic terms:

Proposition 4.12 *Let $\lambda = -\frac{k}{2}$ for some $k \in 2\mathbb{N} + 1$ and $\psi^+ \in \Gamma(S^{+\partial_r}(\bar{X}^+, \bar{g})_{|M})$. Then there exist $\eta \in \mathcal{A}^+$ and $\phi \in \mathcal{A}^-$ such that $r^{\frac{n-k}{2}}\eta + r^{\frac{n-k}{2}+k}\log(r)\phi$ is a solution of equation (4.13) corresponding to the eigenvalue $i\lambda$. Here, η is unique up to $O(r^k)$, and ϕ is unique up to $O(r^1)$, and one has $\eta_{|r=0} = \psi^+$.*

Proof. We make the ansatz $\bar{\theta} = r^{\frac{n-k}{2}}\eta + r^{\frac{n-k}{2}+k}log(r)\phi$, where $\eta = \eta^+ + \eta^-$ and

$\phi = \phi^+ + \phi^-$ for unknown $\eta^\pm, \phi^\pm \in \mathcal{A}^\pm$. The equation we have to solve is

$$\begin{aligned}
&i\lambda \sum_j r^{\mu+j}\eta_j + i\lambda \log(r) \sum_j r^{\mu+k+j}\phi_j \\
=& -\nu \cdot \sum_{j,l} r^{\mu+j+l+1}\tilde{\not{D}}^{h_r,l}\eta_j - \log(r)\nu \cdot \sum_{j,l} r^{\mu+k+j+l+1}\tilde{\not{D}}^{h_r,l}\phi_j \\
&- \frac{n}{2}\nu \cdot \sum_{j,l} r^{\mu+j+l+1}H_r^l \cdot \eta_j - \frac{n}{2}\log(r)\nu \cdot \sum_{j,l} r^{\mu+k+j+l+1}H_r^l \cdot \theta_j \\
&+ \nu \cdot \sum_j (\mu+j) r^{\mu+j}\eta_j + \log(r)\nu \cdot \sum_j (\mu+k+j) r^{\mu+k+j}\phi_j + \nu \cdot \sum_j r^{\mu+k+j}\phi_j \\
&- \frac{n}{2}\nu \cdot \sum_j r^{\mu+j}\eta_j - \frac{n}{2}\log(r)\nu \cdot \sum_j r^{\mu+k+j}\theta_j.
\end{aligned}$$

Analogously to the proof of Proposition 4.10 we can determine η_j for $0 \leq j \leq k-1$ uniquely by ψ^+. The next step is to determine the level k data. The equation we have to look at is

$$i\lambda\eta_k + i\lambda\log(r)\phi_0 = P(\eta_{\leq k-1}) + \frac{k}{2}\nu \cdot \eta_k + \frac{k}{2}\log(r)\nu \cdot \phi_0 + \nu \cdot \phi_0.$$

With respect to the splitting we have:

$$\begin{aligned}
\lambda\eta_k^+ + \lambda\log(r)\phi_0^+ =& \frac{1}{i}P^+(\eta_{\leq k-1}) + \frac{k}{2}\eta_k^+ + \frac{k}{2}\log(r)\phi_0^+ + \phi_0^+, \\
\lambda\eta_k^- + \lambda\log(r)\phi_0^- =& \frac{1}{i}P^-(\eta_{\leq k-1}) - \frac{k}{2}\eta_k^- - \frac{k}{2}\log(r)\phi_0^- - \phi_0^-.
\end{aligned}$$

By the assumption that k is odd, we see that $P^+(\eta_{\leq k-1})$ vanishes since $k-1$ is even. Thus we obtain $-k\eta_k^+ = \phi_0^+ = 0$, $\phi_0^- = \frac{1}{i}P^-(\eta_{\leq k-1})$ and η_k^- remains arbitrary. We set $\eta_k^- := 0$. The determination of η_{k+1} and ϕ_1 is given by the equation

$$\begin{aligned}
i\lambda\eta_{k+1} + i\lambda\log(r)\phi_1 =& P(\eta_{\leq k}) - \nu \cdot \tilde{\not{D}}^{h.0}\phi_0 + \nu \cdot (\frac{k}{2}+1)\eta_{k+1} \\
&+ \nu \cdot \log(r)(\frac{k}{2}+1)\phi_1 + \nu \cdot \phi_1.
\end{aligned}$$

With respect to the splitting we can determine η_{k+1} and ϕ_1 by

$$\begin{aligned}
\phi_1^+ =& \frac{1}{i(k+1)}\tilde{\not{D}}^{h,0}\phi_0^- = -\frac{1}{k+1}\tilde{\not{D}}^{h,0}(P^-(\eta_{\leq k})), \\
\eta_{k+1}^+ =& -\frac{1}{i(k+1)}P^+(\eta_{\leq k}) + \phi_1^+, \\
\eta_{k+1}^- =& \frac{1}{k+1}\phi_1^- = 0.
\end{aligned}$$

This is the new basis of induction. Now let us assume that for $m \geq 2$ we have well determined spinors η_{k+m} and θ_m. To determine η_{k+m+1} and ϕ_{m+1}, we consider the equation

$$\begin{aligned}i\lambda\eta_{k+m+1} + i\lambda\log(r)\phi_{m+1} =& P(\eta_{\leq k+m}) + \log(r)R(\phi_{\leq m})\\ &+ \nu\cdot(\mu+k+m+1)(\eta_{k+m+1}+\log(r)\phi_{m+1})\\ &+ \nu\cdot\phi_{m+1} - \nu\cdot\frac{n}{2}(\eta_{k+m+1}+\log(r)\phi_{m+1}),\end{aligned}$$

for an appropriate operator R. There is no need of an explicit kowledge of this operator, it just has to exist. Again with respect to the splitting we have

$$\begin{aligned}-\frac{k}{2}(\eta^+_{k+m+1}+\log(r)\phi^+_{m+1}) =& \frac{1}{i}P^+(\eta_{\leq k+m}) + \frac{1}{i}\log(r)R^+(\phi_{\leq m}) + \phi^+_{m+1}\\ &+ (\frac{k}{2}+m+1)(\eta^+_{k+m+1}+\log(r)\phi_{m+1})\end{aligned}$$

and

$$\begin{aligned}-\frac{k}{2}(\eta^-_{k+m+1}+\log(r)\phi^-_{m+1}) =& \frac{1}{i}P^-(\eta_{\leq k+m}) + \frac{1}{i}\log(r)R^-(\phi_{\leq m}) - \phi^+_{m+1}\\ &- (\frac{k}{2}+m+1)(\eta^+_{k+m+1}+\log(r)\phi_{m+1}).\end{aligned}$$

These equations determine η_{k+m+1} and ϕ_{m+1} completely by

$$\begin{aligned}\phi^+_{m+1} =& \frac{i}{k+m+1}R^+(\phi_{\leq m}),\\ \eta^+_{k+m+1} =& \frac{i}{k+m+1}P^+(\eta_{\leq k+m}) - \frac{1}{k+m+1}\phi^+_{m+1},\\ \phi^-_{m+1} =& \frac{1}{i(m+1)}R^-(\phi_{\leq m}),\\ \eta^-_{k+m+1} =& \frac{1}{i(m+1)}P^-(\eta_{\leq k+m}) - \frac{1}{m+1}\phi^-_{m+1}.\end{aligned}$$

In case m even, we have $P(\eta_{\leq k+m}) = P^+(\eta_{\leq k+m}) + 0$ and $R(\phi_{\leq m}) = R^+(\phi_{\leq m}) + 0$, whereas in case m odd, we have $P(\eta_{\leq k+m}) = 0 + P^-(\eta_{\leq k+m})$ and $R(\phi_{\leq m}) = 0 + R^-(\phi_{\leq m})$, hence only positive or negative part do arise. This leads to the alternating vanishing of the positive or negative part of our spinors η and ϕ. This completes the induction and finally the proof. □

Remark 4.13 In analogy to Remark 4.11 we can change the assumptions of Proposition 4.12 by taking a negative spinor $\psi^- \in \Gamma(S^{-\partial_r}(\bar{X}^+,\bar{g})_{|M})$, and then we end up with a solution $r^{\frac{n-k}{2}}\eta + r^{\frac{n+k}{2}}\log(r)\phi$ of the eigenequation with eigenvalue $-i\lambda$, such that $\eta \in \mathcal{A}^-$ is unique up to $O(r^k)$ and $\phi \in \mathcal{A}^+$ is unique up to $O(r^1)$, and one has $\eta_{|r=0} = \psi^-$.

The failure of uniquness in Proposition 4.12 and in Remark 4.13 is due to the fact that we have randomly set at the critical step η_k^- to zero. From this observation we can define an operator depending on the eigenvalue $\lambda = -\frac{k}{2}$ for odd $k \in \mathbb{N}$ by

$$\begin{aligned}\mathcal{T}_\pm(\lambda) : \Gamma\big(S^{\pm\partial_r}(\bar{X}^+, \bar{g})_{|M}\big) &\to \Gamma\big(S^{\mp\partial_r}(\bar{X}^+, \bar{g})_{|M}\big)\\ \psi^\pm &\mapsto (\phi^\pm)_{|M},\end{aligned}$$

where $\phi_\pm$ are the two log $-$coefficients of the solutions $\bar{\theta}_+$ from Proposition 4.12 induced by ψ^+, and $\bar{\theta}_-$ from Remark 4.13 induced by ψ^-. Note that $\phi^\pm$ restricted to $r=0$ are uniquely determined by $\psi^\pm$. The construction of these solutions implies that $\mathcal{T}_\pm(\lambda)$ is a $h-$geometric differential operator of order k with leading term $(\tilde{\not{D}}^{h,0})^k$. Let us recall the splitting of the spinor bundle $S(\bar{X}^+, \bar{g})$ restricted to the boundary

$$S^{+\partial_r}(\bar{X}^+, \bar{g})_{|M} \oplus S^{-\partial_r}(\bar{X}^+, \bar{g})_{|M},$$

and let us introduce the projection maps $p^\pm(\psi) := \frac{1}{2}(\psi \mp i\partial_r \cdot \psi)$ for $\psi \in S(\bar{X}^+, \bar{g})_{|M}$. Thus we may define for $k \in \mathbb{N}$ odd and $\lambda = -\frac{k}{2}$ an $h-$geometric differential operator:

$$\begin{aligned}\bar{\mathcal{T}}(\lambda; h) : \Gamma\big(S(\bar{X}^+, \bar{g})_{|M}\big) &\to \Gamma\big(S(\bar{X}, \bar{g})_{|M}\big)\\ \psi &\mapsto \bar{\mathcal{T}}(\lambda; h)\psi := \mathcal{T}_-\big(p^-(\psi)\big) + \mathcal{T}_+\big(p^+(\psi)\big).\end{aligned}$$

Now let us state a corollary concerning the structure of $\bar{\mathcal{T}}(\lambda; h)$ for $\lambda = -\frac{k}{2}$, where $k \in \mathbb{N}$ is odd.

Corollary 4.14 *One has $\bar{\mathcal{T}}(\lambda; h) = -\nu \cdot \bar{P}_k(h)$ for some $h-$geometric differential operators $\bar{P}_k(h) : \Gamma\big(S(\bar{X}^+, \bar{g})_{|M}\big) \to \Gamma\big(S(\bar{X}, \bar{g})_{|M}\big)$.*

Proof. This is a consequence of the linearity of the construction performed in the proof of Proposition 4.10, especially equation (4.15). □

4.4 Conformally covariant differential operators hidden in formal eigenspinors

This section aims to show in which sense conformally covariant differential operators appear inside the formal asymptotics constructed in Section 4.3, and to specify these operators. For odd $k \in \mathbb{N}$ and $\lambda = -\frac{k}{2}$ we first state a conformal covariance property of the operators $\bar{\mathcal{T}}(\lambda; h)$, and, by construction, this property extends to the operators $\bar{P}_k$, given in Corollary 4.14. Identifying the spinor bundle $S(M, h)$ inside $S(\bar{X}^+, \bar{g})_{|M}$ leads to an $h-$geometric conformally covariant differential operators acting on $S(M, h)$ with leading term $\not{D}^k$. This was also done in [GMP10, Lemma 4.10] and [GMP12, Corollary 22].

First recall that the operator $\bar{\mathcal{T}}(\lambda; h)$ for $\lambda = -\frac{k}{2}$ acts on spinor fields defined on the boundary of $(\bar{X}^+, \bar{g})$. All the constructions made in the last Section depend on a chosen $h \in c$. Changing the metric h to $\hat{h} := e^{2\sigma} h$ leads to a new defining function $\hat{r} := e^{\sigma} r$.

Lemma 4.15 *For conformally equivalent metrics h and $\hat{h} = e^{2\sigma} h$, the operators $\bar{\mathcal{T}}(\lambda; h)$ and $\bar{\mathcal{T}}(\lambda; \hat{h})$ for $\lambda = -\frac{k}{2}$ are related by*

$$e^{\frac{n+k}{2}\sigma} \bar{\mathcal{T}}(\lambda; \hat{h})\psi = \bar{\mathcal{T}}(\lambda; h)(e^{\frac{n-k}{2}\sigma}\psi),$$

for all $\psi \in \Gamma\big(S(\bar{X}^+, \bar{g})_{|M}\big)$.

Proof. Let $\psi \in \Gamma\big(S(\bar{X}^+, \bar{g})_{|M}\big)$ and consider its decomposition $\psi = p^+(\psi) + p^-(\psi)$. The spinor construction we have performed in Proposition 4.12 relies on the choice of an element h of the conformal class. Let us denote by $\theta := \theta_+ + \theta_- = \hat{\theta}_+ + \hat{\theta}_-$ the spinor we get from Proposition 4.12 and Remark 4.13 with respect to the defining functions r and $\hat{r} = e^{\sigma} r$, respectively. From the detailed representations

$$\begin{aligned} \theta_+ &= r^{\frac{n-k}{2}} \eta_+ + \log(r) r^{\frac{n+k}{2}} \phi_+, \\ \theta_- &= r^{\frac{n-k}{2}} \eta_- + \log(r) r^{\frac{n+k}{2}} \phi_- \end{aligned}$$

with respect to r, and

$$\begin{aligned} \hat{\theta}_+ &= \hat{r}^{\frac{n-k}{2}} \hat{\eta}_+ + \log(\hat{r}) \hat{r}^{\frac{n+k}{2}} \hat{\phi}_+, \\ \hat{\eta}_- &= \hat{r}^{\frac{n-k}{2}} \hat{\eta}_- + \log(\hat{r}) \hat{r}^{\frac{n+k}{2}} \hat{\phi}_- \end{aligned}$$

with respect to $\hat{r}$, we get the following identities by the uniqueness property:

$$\begin{aligned} (\eta_\pm)_{|M} &= e^{\frac{n-k}{2}\sigma} (\hat{\eta}_\pm)_{|M}, \\ (\phi_\pm)_{|M} &= e^{\frac{n+k}{2}\sigma} (\hat{\phi}_\pm)_{|M}. \end{aligned}$$

On the one hand we have

$$\bar{\mathcal{T}}(\lambda; \hat{h})(\hat{\eta}_{|M}) = \mathcal{T}(\lambda; \hat{h})(e^{-\frac{n-k}{2}\sigma} \eta_{|M}),$$

and on the other hand we have

$$\begin{aligned} \bar{\mathcal{T}}(\lambda; \hat{h})(\hat{\eta}_{|M}) =& \hat{\phi}_{|M} = e^{-\frac{n+k}{2}\sigma} \phi_{|M} \\ =& e^{-\frac{n+k}{2}\sigma} \bar{\mathcal{T}}(\lambda; h)(\eta_{|M}). \end{aligned}$$

Now $\eta_{|M} = \psi$ completes the proof the lemma. ☐

The conformal covariance of the operator $\bar{\mathcal{T}}(\lambda; h)$, for $\lambda = -\frac{k}{2}$, directly gives:

Corollary 4.16 *Let $\psi \in \Gamma(S(\bar{X}^+, \bar{g})_{|M})$ and $k \in \mathbb{N}$ be odd. Then one has*

$$e^{\frac{n+k}{2}\sigma} \bar{P}_k(\hat{h})(\psi) = \bar{P}_k(h)(e^{\frac{n-k}{2}\sigma}\psi).$$

Proof. Because of $\bar{\mathcal{T}}(-\frac{k}{2}; h) = -\nu \cdot \bar{P}_k(h)$, the corollary follows from $(r\partial_r)_{|M} = (\hat{r}\partial_{\hat{r}})_{|M}$, since $r\partial_r$ and $\hat{r}\partial_{\hat{r}}$ have the same lenght and direction when restricted to $r = 0$. $\square$

Now we show how the $\bar{P}_k(h)$'s induce $h-$geometric differential operators acting on $S(M^n, h)$ which are conformally covariant of certain bi-degrees. Let us denote the isomorphism (4.1) by $\iota^h : \tilde{S}(M, h) \to S(\bar{X}^+, \bar{g})_{|M}$, where

$$\tilde{S}(M, h) := \begin{cases} S(M, h), & n = 2m \\ S(M, h) \oplus S(M, h), & n = 2m + 1 \end{cases},$$

and define

$$\begin{aligned} P_k(h) : \Gamma\big(\tilde{S}(M, h)\big) &\to \Gamma\big(\tilde{S}(M, h)\big) \\ \varphi &\mapsto P_k(h)(\varphi) := (\iota^h)^{-1} \circ \bar{P}_k(h)\big(\iota^h(\varphi)\big). \end{aligned}$$

The $P_k(h)$'s are $h-$geometric differential operators and contain exactly what we are looking for: a conformal kth power of the Dirac operator on semi Riemannian spin manifolds. This was obtained by the authors in [GMP12, Corollary 22] for Riemannian manifolds, but their method works also for arbitrary spin manifolds.

Theorem 4.17 *Let (M, h) be a spin manifold. For all odd $k \in \mathbb{N}$ there exists a conformally covariant operator $\mathcal{D}_k : \Gamma\big(S(M, h)\big) \to \Gamma\big(S(M, h)\big)$ of bi-degree $(\frac{k-n}{2}, -\frac{k+n}{2})$ with leading term $\not{D}^k$. In case of even dimensional manifolds we only have existence up to $k \leq n$.*

Proof. Let the dimension of M be even. In this case, Corollary 4.16 shows that $P_k(h)$ defines a conformally covariant differential operator with right bi-degree and, after normalisation, with leading term $\not{D}^k$. For M odd dimensional, the isomorphism ι^h implies that $P_k(h)$ restricted to $\Gamma\big(S(M, h)\big)$ has target space $\Gamma\big(S(M, h)\big)$, and that this restriction is a conformally covariant differential operator of the right bi-degree. After normalisation, it has leading term $\not{D}^k$. In both cases we denote the resulting operator by $\mathcal{D}_k$. In case of even dimensional manifolds the Poincaré metric is uniqly determined up to order r^{n-2}, hence we get existence of $\mathcal{D}_k$ up to $k \leq n$ only. $\square$

The operators $\mathcal{D}_k$ for odd $k \in \mathbb{N}$ are conformal kth powers of the Dirac operator. Since the proof of its existence was constructive, we can compute the first powers. This was also achieved in [GMP12, Theorem 23] by a direct computation, but without an explicit use of the variation of the Dirac operator with respect to h_r.

Proposition 4.18 *Let (M^n, h) be a spin manifold. The conformal first and third power of the Dirac operator is given by*

$$\begin{aligned}\mathcal{D}_1 =& \not{D},\\ \mathcal{D}_3 =& \not{D}^3 - (P, \nabla^{S(M,h)}) - (\nabla^{S(M,h)}, P\cdot),\end{aligned}$$

where P is the Schouten tensor of h, and the bracket notations were introduced in the equations (2.4) *and* (2.7)*.*

Proof. Let $\psi \in \Gamma(S(M,h))$. In order to compute $\mathcal{D}_1$ and $\mathcal{D}_3$, note that the operators we are looking for sit in the spinors $\theta_\pm$ given in Proposition 4.10 and Remark 4.11 for the given $\psi^\pm := p^\pm(\iota(\varphi)) \in \Gamma(S^{\pm\partial_r}(\bar{X}^+, \bar{g})_{|M})$. The first equations arising through the eigenequation are

$$\begin{aligned}
0 =& \pm i\lambda(\theta_\pm)_0 - \nu \cdot \lambda(\theta_\pm)_0\\
0 =& \pm i\lambda(\theta_\pm)_1 - \nu \cdot (\lambda+1)(\theta_\pm)_1 + \nu \cdot \tilde{\not{D}}^{h,0}(\theta_\pm)_0 + \nu \cdot \frac{n}{2} H_0^0(\theta_\pm)_0\\
0 =& \pm i\lambda(\theta_\pm)_2 - \nu \cdot (\lambda+2)(\theta_\pm)_2 + \nu \cdot (\tilde{\not{D}}^{h,0}(\theta_\pm)_1 + \tilde{\not{D}}^{h,1}(\theta_\pm)_0)\\
&+ \nu \cdot \frac{n}{2}(H_0^0(\theta_\pm)_1 + H_0^1(\theta_\pm)_0)\\
0 =& \pm i\lambda(\theta_\pm)_3 - \nu \cdot (\lambda+2)(\theta_\pm)_3 + \nu \cdot (\tilde{\not{D}}^{h,0}(\theta_\pm)_2 + \tilde{\not{D}}^{h,1}(\theta_\pm)_1 + \frac{1}{2}\tilde{\not{D}}^{h,2}(\theta_\pm)_0)\\
&+ \nu \cdot \frac{n}{2}(H_0^0(\theta_\pm)_2 + H_0^1(\theta_\pm)_1 + \frac{1}{2} H_0^2(\theta_\pm)_0).
\end{aligned}$$

These equations become simpler as the formal power series of the operators $\tilde{\not{D}}^{h_r}$ has vanishing odd summands, whereas the formal power series of H_r has vanishing even summands. In what follows, we shall implicitely restrict everything to the boundary. The first equation told us $((\theta_\pm)_0^\pm)_{|M} = \psi^\pm$ and $(\theta_\pm)_0^\mp = 0$. Now, $\mathcal{D}_1 = \not{D}$ follows from

$$\begin{aligned}
(\theta_+)_1^+ = 0, \quad (\theta_+)_1^- = \frac{1}{2\lambda+1}\tilde{\not{D}}^{h,0}(\theta_+)_0^+\\
(\theta_-)_1^- = 0, \quad (\theta_-)_1^+ = \frac{1}{2\lambda+1}\tilde{\not{D}}^{h,0}(\theta_-)_0^-,
\end{aligned}$$

since $((\theta_+)_1 + (\theta_-)_1)_{|M} = \frac{1}{2\lambda+1}\tilde{\not{D}}^{h,0}(\psi^+ + \psi^-)$, and we identify $\not{D}$ at the pole $\lambda = -\frac{1}{2}$. In order to compute $\mathcal{D}_3$, we need the third equation, which shows

$$(\theta_+)_2^- = 0, \quad (\theta_+)_2^+ = \frac{1}{2}(\tilde{\not{D}}^{h,0}(\theta_+)_1^- + \frac{1}{2} J(\theta_+)_0^+)$$

$$(\theta_-)_2^+ = 0, \quad (\theta_+)_2^- = \frac{1}{2}(\tilde{\not{D}}^{h,0}(\theta_-)_1^+ + \frac{1}{2}J(\theta_-)_0^-).$$

Hence the fourth equation gives us

$$\begin{aligned}
(\theta_+)_3^+ = 0, \quad (\theta_+)_3^- &= \frac{1}{2\lambda+3}[\tilde{\not{D}}^{h,0}(\theta_+)_2^+ + \frac{1}{2}\tilde{\not{D}}^{h,2}(\theta_+)_0^+ + \frac{1}{2}J(\theta_+)_1^-] \\
(\theta_-)_3^- = 0, \quad (\theta_-)_3^+ &= \frac{1}{2\lambda+3}[\tilde{\not{D}}^{h,0}(\theta_-)_2^- + \frac{1}{2}\tilde{\not{D}}^{h,2}(\theta_-)_0^- + \frac{1}{2}J(\theta_-)_1^+].
\end{aligned}$$

Therefore, we are able to compute $(\theta_\pm)_3$ in terms of $\psi^\pm = p^\pm(\iota(\psi)) = (\theta_\pm)_0$:

$$\begin{aligned}
((\theta_+)_3 + (\theta_-)_3)_{|M} =& \frac{1}{2\lambda+3}\Big[\frac{1}{2}\frac{1}{2\lambda+1}(\tilde{\not{D}}^{h,0})^3(\psi^+ + \psi^-) + \frac{1}{4}\tilde{\not{D}}^{h,0}(J(\psi^+ + \psi^-)) \\
&+ \frac{1}{2}\tilde{\not{D}}^{h,2}(\psi^+ + \psi^-) + \frac{1}{2}\frac{1}{2\lambda+1}J\tilde{\not{D}}^{h,0}(\psi^+ + \psi^-)\Big].
\end{aligned}$$

The operator $\mathcal{D}_3$ sits at the pole $\lambda = -\frac{3}{2}$, hence, dropping the factor $-\frac{1}{4}$ we obtain

$$\begin{aligned}
\mathcal{D}_3 =& \not{D}^3 - 2(P, \nabla) + \mathrm{grad}^g(J)\cdot \\
=& \not{D}^3 - (P, \nabla^{S(M,h)}) - (\nabla^{S(M,h)}, P\cdot),
\end{aligned}$$

where we have made use of the second metric variation of the Dirac operator with respect to h_r, see Example 4.5. This completes the proof. □

In principle, one can compute conformal higher powers of the Dirac operator in this setting. But the implicit form of the Poincaré metric and the higher variations of Dirac operators spoils the task of computation.

5 Construction of conformally covariant differential operators via the tractor machinery

5.1 The standard tractor bundle

Let (M, c) be a conformal manifold. In Section 3.5 we have associate to (M, c) a Cartan geometry $(\mathcal{P}^1, w^{nc})$ of type (G, B). Considering the standard representation $\rho : O(p+1, q+1) \to GL(\mathbb{R}^{n+2})$ leads to the standard tractor bundle of (M, c):

$$\mathcal{T}(M) := \mathcal{P}^1 \times_{(B,\rho)} \mathbb{R}^{p+1,q+1}.$$

The normal conformal Cartan connection w^{nc} on $\mathcal{P}^1$ induces, by Lemma 2.9, a covariant derivative $\nabla^{\mathcal{T}(M)}$ on $\mathcal{T}(M)$. Furthermore, the pseudo Riemannian metric $\langle \cdot, \cdot \rangle_{p+1,q+1}$ on $\mathbb{R}^{n+2}$ defines a bundle metric $g^{\mathcal{T}}(t_1, t_2) := \langle v_1, v_2 \rangle_{p+1,q+1}$ for $t_i = [H, v_i] \in \mathcal{T}(M)$, $i = 1, 2$, which is parallel with respect to $\nabla^{\mathcal{T}(M)}$. This is well defined, since the metric $\langle \cdot, \cdot \rangle_{p+1,q+1}$ is $B-$invariant.

If $g \in c$, then $(\mathcal{P}^g, \pi, M, SO(p, q))$ is a $\iota^0 : SO(p, q) \to CO(p, q)-$reduction of the $\mathcal{P}^0$, and also a $\iota : SO(p, q) \to B-$reduction of $\mathcal{P}^1$. These reductions yield the isomorphisms

$$\begin{aligned}
\mathcal{T}(M) &\simeq \mathcal{P}^g \times_{(O(p,q),\rho)} \mathbb{R}^{n+2}, \\
TM &\simeq \mathcal{P}^g \times_{(O(p,q),Ad)} \mathfrak{b}_{-1}, \\
T^*M &\simeq \mathcal{P}^g \times_{(O(p,q),Ad)} \mathfrak{b}_1, \\
\mathfrak{so}(TM, g) &\simeq \mathcal{P}^g \times_{(O(p,q),Ad)} \mathfrak{so}(p, q).
\end{aligned}$$

Let us define an action $\rho^g : V \to End(\mathcal{T}(M))$, where V is any of the bundles TM, T^*M or $\mathfrak{so}(TM, g)$, by

$$\rho^g(\Theta)t := [s, \rho_*([u]^{-1}(\Theta))v],$$

where $t = [s, v] \in \mathcal{P}^g \times_{(O(p,q),\rho)} \mathbb{R}^{n+2}$ for $s \in \mathcal{P}^g$ and $v \in \mathbb{R}^{n+1}$, $\Theta \in V$, and $[s] : W \to V$ is the isomorphism given by $[s]w := [s, w]$ for $w \in W = \mathfrak{b}_{-1}, \mathfrak{b}_1, \mathfrak{so}(p, q)$, respectively.

Lemma 5.1 *The covariant derivative $\nabla^{\mathcal{T}(M)}$ induced by the normal conformal Cartan connection is given with respect to a metric $g \in c$ by*

$$\nabla_X^{\mathcal{T}(M)} t = \nabla_X^g t + \rho^g(X)t - \rho^g(P^g(X))t, \tag{5.1}$$

where $t \in \Gamma(\mathcal{T}(M))$, $X \in \mathfrak{X}(M)$, $P^g \in \Omega^1(M, T^*M)$ *and* ∇^g *is the associated covariant derivative induced by the Levi-Civita connection.*

Proof. Using the reductions mentioned above, we can represent t, locally, as $t = [s, v] = [\sigma^{A^g}(s), v]$ for a local section $s : U \to \mathcal{P}^g$, a smooth map $v : U \to \mathbb{R}^{n+2}$ and the B_0-equivariant section $\sigma^{A^g} : \mathcal{P}^g \to \mathcal{P}^1$ induced by the Leci-Civita connection. The proof follows form equation (3.11) in the following way,

$$\begin{aligned}
\nabla_X^{\mathcal{T}(M)} t &= \Big[s, X(v) + \rho_*\Big(w^{nc}\big(d(\sigma^{A^g} \circ s)(X)\big)\Big)v\Big] \\
&= \left[s, X(v) + \rho_*\Big(\theta^{\mathcal{P}^0}(ds(X)) + A^g(ds(X)) - \sum_i P^g(X, s_i)e_i^*\Big)v\right] \\
&= \Big[s, X(v) + \rho_*\Big(A^g(ds(X))\Big)v\Big] + \Big[s, \rho_*([s]^{-1}X)v\Big] - \Big[s, \rho_*([s]^{-1}P^g(X))v\Big] \\
&= \nabla_X^g t + \rho^g(X)t - \rho^g(P^g(X))t,
\end{aligned}$$

where $X \in \mathfrak{X}(M)$. □

Remark 5.2 All we have done above with respect to the standard representation ρ of $O(p+1, q+1)$ could also be carried out with an arbitrary finite dimensional representation.

Another consequence of making a choice $g \in c$ is the splitting of the standard tractor bundle into a direct sum of vector bundles.

Lemma 5.3 *For any* $g \in c$, *there exists a vector bundle isomorphism*

$$\Phi^g : \mathcal{T}(M) \to \underline{M} \oplus TM \oplus \underline{M} =: \mathcal{T}(M)_g$$

where $\underline{M} := M \times \mathbb{R}$ *is the trivial bundle.*

Proof. Consider $t = [s, v]$ for a local section $s : U \to \mathcal{P}^g$ and a smooth map $v : U \to \mathbb{R}^{n+2}$. Representing the vector $v = (\alpha, x = (x_1, \ldots, x_n), \beta) \in \mathbb{R} \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}$ with respect to the basis $(f_0, e_1, \ldots, e_n, f_{n+1})$, we may define the bundle isomorphism $\Phi^g(t) := (\alpha, X, \beta)$, where $X := [s]^{-1}(x) \in TM$. □

We call Φ^g the $g-$trivialization of the standard tractor bundle. The essential property of Φ^g is that all tractor objects which only depend on the conformal structure can be represented in terms of the underlying metric g by compositions with Φ^g and its inverse. This will be referred to as the $g-$metric representation of the corresponding tractor object.

For $X \in \mathfrak{X}(M)$ the $g-$metric representation of the covariant derivative $\nabla^{\mathcal{T}(M)}$ is given by

$$\nabla_X^{g,\mathcal{T}(M)} : \Gamma(\mathcal{T}(M)_g) \to \Gamma(\mathcal{T}(M)_g)$$

$$t_g \mapsto \nabla_X^{g,\mathcal{T}(M)}(t_g) := \Phi^g \circ \nabla_X^{\mathcal{T}(M)}\big((\Phi^g)^{-1}(t_g)\big).$$

Then we may compute:

Lemma 5.4 *Let $t_g = (\alpha, Y, \beta) \in \Gamma(\mathcal{T}(M)_g)$. Then one has*

$$\nabla_X^{g,\mathcal{T}(M)} \begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix} = \begin{pmatrix} \nabla_X^{LC} & -P^g(X,\cdot) & 0 \\ X & \nabla_X^{LC} & P^g(X)^\natural \\ 0 & -g(X,\cdot) & \nabla_X^{LC} \end{pmatrix} \begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix},$$

where P^g is the Schouten tensor and ∇^{LC} the Levi-Civita connection, both with respect to g.

Proof. In view of Lemma 5.1, we have to compute the three summands of equation (5.1) in terms of the standard representation. Fix $\psi \in \Gamma(\mathcal{T}(M))$ and $X \in \mathfrak{X}(M)$ arbitrarily. With respect to a local section $u \in \Gamma(\mathcal{P}^g)$ we have $\psi = [u, v] \in \mathcal{T}(M)$ and $X = [u, x] \in TM$. The first summand from $\nabla^{g,\mathcal{T}(M)}$ is given by

$$\begin{aligned} \nabla_X^g \psi &= \Big[u, X(v) + \rho_*\Big(A^g(du(X))\Big)v\Big] \\ &= [u, X(v)] + \sum_{i<j} \varepsilon_i \varepsilon_j \Big[u, M(0, (0, g(\nabla_X^{LC} s_i, s_j)), 0)v\Big], \end{aligned}$$

since ρ_* acts as the identity as ρ is the standard representation. The second summand yields

$$\begin{aligned} \rho^g(X)\psi &= [u, \rho_*(M(x, (0,0), 0))v] \\ &= [u, M(x, (0,0), 0)v]. \end{aligned}$$

The last summand simplifies to

$$\rho^g(P(X))\psi = [u, M(0, (0,0), z)v],$$

where $T^*M \ni P(X) = [u, z]$ for $z \in (\mathbb{R}^{p,q})^*$. With respect to the $g-$trivialization, Lemma 5.3, the endomorphisms $\rho^g(\cdot) \in End(\mathcal{T}(M), \mathcal{T}(M))$ act by

$$\begin{aligned} \rho^g(X)(\alpha, Y, \beta) &= (0, \alpha X, -g(X, Y)), \\ \rho^g(\mu)(\alpha, Y, \beta) &= (\mu(Y), -\beta\mu^\natural, 0), \\ \rho^g(\tau)(\alpha, Y, \beta) &= (0, \tau(Y), 0), \end{aligned}$$

where $X \in TM$, $\mu \in T^*M$ and $\tau \in \mathfrak{so}(M, g)$, which proves the lemma. $\square$

The following lemma shows how we can identify $\mathcal{T}(M)_g$ with respect to different $g \in c$.

Lemma 5.5 *Let $g, \hat{g} := e^{2\sigma} g \in c$. The map $T(g,\sigma) := \Phi^{\hat{g}} \circ (\Phi^g)^{-1} : \mathcal{T}(M)_g \to \mathcal{T}(M)_{\hat{g}}$ is given by the matrix*

$$T(g,\sigma) = \begin{pmatrix} e^{-\sigma} & -e^{-\sigma}\operatorname{grad}^g(\sigma)^\flat & -\frac{1}{2}e^{-\sigma}|d\sigma|_g^2 \\ 0 & e^{-\sigma} & e^{-\sigma}\operatorname{grad}^g(\sigma) \\ 0 & 0 & e^{\sigma} \end{pmatrix}.$$

Proof. Let $g \in c$ and $\hat{g} = e^{2\sigma} g \in c$ with their Levi-Civita connections A^g and $A^{\hat{g}}$, viewed as connection 1−forms on the orthonormal frame bundles $\mathcal{P}^g$, and, respectively, $\mathcal{P}^{\hat{g}}$, be given. The choice $g \in c$ and the inclusion $SO(p,q) \subset CO(p,q) \simeq B_0$ determines reductions of the conformal frame bundle to the $g-$frame bundle $\mathcal{P}^g$ of (M,g), respectively $\mathcal{P}^{\hat{g}}$ of $(M,\hat{g})$. Let us denote the reduction maps by $\iota : \mathcal{P}^g \to \mathcal{P}^0$ and $\hat{\iota} : \mathcal{P}^{\hat{g}} \to \mathcal{P}^0$, respectively. Thus, there exist principal isomorphisms

$$\mathcal{P}^g \times_{SO(p,q)} B_0 \simeq \mathcal{P}^0 \simeq \mathcal{P}^{\hat{g}} \times_{SO(p,q)} B_0.$$

Since the calculations will be local, let $\gamma : I \to U \subset M$ be a smooth curve, and let $s : U \to \mathcal{P}^g$ and $\hat{s} : U \to \mathcal{P}^{\hat{g}}$ be local sections. Furthermore, for $p \in \mathcal{P}^0$ and $Y \in T_p\mathcal{P}^0$, we consider a curve $\delta : I \to \mathcal{P}^0$ such that $\delta(0) = p$ and $Y = \dot{\delta}(0)$. Equivalently, for $s(t) := s \circ \gamma(t)$ and $\hat{s}(t) := \hat{s} \circ \gamma(t)$ we have $\delta(t) = [s(t), b(t)]$ with respect to g, or $\delta(t) = [\hat{s}(t), \hat{b}(t)]$ with respect to $\hat{g}$. The local section s and $\hat{s}$ are also local sections of $\mathcal{P}^0$, hence they differ by some $a : I \to B_0$, i.e., $a(t) = (e^{-\sigma\circ\gamma(t)}, I_n)$ and $\hat{s}(t) = s(t)\cdot a(t)$. Consequently, we have $\hat{b}(t) = a^{-1}(t)\cdot b(t)$. Let us extend the connection forms A^g and $A^{\hat{g}}$ to the conformal frame bundle $\mathcal{P}^0$. These extensions will be denoted by γ^g and $\gamma^{\hat{g}}$, and they are given by

$$\begin{aligned}\gamma_p^g(Y) :=& Ad(b(0)^{-1})(\pi_{\mathcal{P}^g}^* A^g)_{[s(0),b(0)]}(\dot{\delta}(0)) + (\pi_{B_0}^* w_{B_0})_{[s(0),b(0)]}(\dot{\delta}(0)) \\ =& Ad(b(0)^{-1}) A^g_{s(0)}(\dot{s}(0)) + (w_{B_0})_{b(0)}(\dot{b}(0)),\end{aligned}$$

where $(\pi_{\mathcal{P}^g})_{[s,b]}$ and $(\pi_{B_0})_{[s,b]}$ are the projections to $\mathcal{P}^g$ and B_0, and w_{B_0} is the Maurer-Cartan form of B_0. Note that $\gamma^{\hat{g}}$ is defined analogously. Furthermore, let us recall the relation of the corresponding Levi-Civita connections $\nabla_X^{\hat{g}} Y = \nabla_X^g Y + X(\sigma)Y + Y(\sigma)X - g(X,Y)\operatorname{grad}^g(\sigma)$. Thus we get

$$\begin{aligned}\hat{g}(\hat{\nabla}_X \hat{s}_i, \hat{s}_j) =& g(-X(\sigma)s_i + \nabla_X s_i + X(\sigma)s_i + s_i(\sigma)X - g(X,s_i)\operatorname{grad}^g(\sigma), s_j) \\ =& g(\nabla_X s_i, s_j) + \varepsilon_j s_i(\sigma)X^j - \varepsilon_i X^i s_j(\sigma),\end{aligned}$$

where $X = \sum_i X^i s_i$. For $\dot{\gamma}(0) = \sum \dot{\gamma}(0)^i s_i$ this leads to

$$A^{\hat{g}}_{\hat{s}(0)}(\dot{\hat{s}}(0)) = A^g_{s(0)}(\dot{s}(0)) + \sum_{i<j}(\varepsilon_i s_i(\sigma)\dot{\gamma}(0)^j - \varepsilon_j s_j(\sigma)\dot{\gamma}(0)^i)E_{ij}.$$

Let us denote the last summand from the last equation by $\Xi(\dot{\gamma}(0))$. From

$$\dot{\hat{b}}(0) = dR_{b(0)}(\dot{a^{-1}}(0)) + dL_{b(0)^{-1}}\widetilde{(\dot{b}(0))(a^{-1}(0)b(0))}$$

we get

$$(w_{B_0})_{\hat{b}(0)}(\dot{\hat{b}}(0)) = Ad(b(0)^{-1})(w_{B_0})_{a^{-1}(0)}(\dot{a^{-1}}(0)) + dL_{b(0)^{-1}}(\dot{b}(0)).$$

Putting things together yields

$$\begin{aligned}\gamma_p^{\hat{g}}(Y) =& \gamma_p^{\hat{g}}(\dot{\delta}(0)) = Ad(\hat{b}^{-1}(0))A^{\hat{g}}_{\hat{s}(0)}(\dot{\hat{s}}(0)) + (w_{B_0})_{\hat{b}(0)}(\dot{\hat{b}}(0)) \\ =& \gamma_p^{g}(Y) + Ad(b(0)^{-1})\Big(\Xi(\dot{\gamma}(0)) + (w_{B_0})_{a^{-1}(0)}(\dot{a^{-1}}(0))\Big),\end{aligned}$$

where we have used $Ad(a(0)) = id$. Furthermore, note that

$$dL_{a(0)}(\dot{a^{-1}}(0)) = d\sigma_{\gamma(0)}(\dot{\gamma}(0))\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Thus, we have accomplished that the difference between the induced connection forms on the conformal frame bundle differ by an element of $\mathfrak{b}_0$. Hence, their difference $\gamma_p^g(Y) - \gamma_p^{\hat{g}}(Y)$ is given by

$$Ad(b(0)^{-1})\Big(\Xi(\dot{\gamma}(0)) + (w_{B_0})_{a(0)^{-1}}(\dot{a^{-1}}(0))\Big) = [Z, \theta_p^{\mathcal{P}^0}(Y)]$$

for $Z := M(0,(0,0),[p]^{-1}d\sigma) \in \mathfrak{b}_1$ and $\theta_p^{\mathcal{P}^0}(Y) = [p]^{-1}d\pi_p^0(Y) = M(\dot{\gamma}(0),(0,0),0) \in \mathfrak{b}_{-1}$, where π^0 is the projection of $\mathcal{P}^0$ to M. This yields

$$(\gamma^g - \gamma^{\hat{g}})_p(Y) = [Z, \theta_p^{\mathcal{P}^0}(Y)].$$

In terms of the mappings $\sigma^{A^g} := \ker(\iota^*\gamma^g) : \mathcal{P}^g \to \mathcal{P}^1$ and $\sigma^{A^{\hat{g}}} := \ker(\hat{\iota}^*\gamma^{\hat{g}}) : \mathcal{P}^{\hat{g}} \to \mathcal{P}^1$, we have $\sigma^{A^g}(s) = \sigma^{A^{\hat{g}}}(\hat{s}) \cdot a(0)^{-1} \cdot exp(-Z) =: \sigma^{\hat{g}}(\hat{s}) \cdot b$ for $b \in B$. The element b is given by the matrix

$$b = \begin{pmatrix} e^{-\sigma(x)} & -e^{-\sigma(x)}[s]^{-1}d\sigma_x & -\frac{1}{2}e^{-\sigma(x)}|d\sigma|^2 \\ 0 & I_n & J^{p,q}[s]^{-1}\operatorname{grad}^g(\sigma)(x) \\ 0 & 0 & e^{\sigma(x)} \end{pmatrix}.$$

An element of the standard tractor bundle $t_x \in \mathcal{T}(M)_x$ can be represented on the one hand by

$$t_x = [\sigma^{A^g}(s),(a,X,b)] \mapsto (a, \sum_i X^i s_i, b) \in \mathbb{R},$$

and on the other hand by

$$t_x = [\sigma^{A^{\hat{g}}}(\hat{s}), (\hat{a}, \hat{X}, \hat{b})] \mapsto (\hat{a}, \sum_i \hat{X}^i \hat{s}_i, \hat{b}).$$

The relation among the vectors (a, X, b) and $(\hat{a}, \hat{X}, \hat{b})$ is given by the matrix

$$\begin{pmatrix} e^{-\sigma(x)} & -e^{-\sigma} d\sigma_x & -\frac{1}{2} e^{-\sigma(x)} |d\sigma_x|_g^2 \\ 0 & e^{-\sigma} I_n & e^{-\sigma(x)} \operatorname{grad}^g(\sigma) \\ 0 & 0 & e^{\sigma(x)} \end{pmatrix},$$

which proves the lemma. □

Summarizing, for a conformal structure (M, c) we have associated a vector bundle $\mathcal{T}(M)$ and a distinguished covariant derivative $\nabla^{\mathcal{T}(M)}$, which is an analogue of (TM, ∇^{LC}) associated to a semi Riemannian structure.

5.2 The standard spin tractor bundle

Let (M, c) be a conformal spin manifold and consider a conformal spin structure $(\mathcal{Q}^0, f^0)$ on (M, c). As we have seen in Section 3.6, the latter possesses a first prolongation $(\mathcal{Q}^1, \pi, M, \tilde{B})$ equipped with the normal conformal spin connection $\tilde{w}^{nc}$. The spin representation

$$\tilde{\rho} := \kappa_{p+1,q+1} : Spin_0(p+1, q+1) \to GL(\Delta_{p+1,q+1})$$

induces the associated vector bundle $\mathcal{S}(M) := \mathcal{Q}^1 \times_{(\tilde{B},\tilde{\rho})} \Delta_{p+1,q+1}$, called the standard spin tractor bundle of (M, c). Note that in what follows, we will denote a section of $\mathcal{P}^g$ and and a section of $\mathcal{S}(M)$ with the same letter s, but their meaning is always clear from the context.

Again, on the standard spin tractor bundle exists a covariant derivative $\nabla^{\mathcal{S}(M)}$ induced by the normal conformal spin connection $\tilde{w}^{nc}$. The $Spin_0(p+1, q+1)-$invariant Hermitian scalar product $< \cdot, \cdot >$ on $\Delta_{p+1,q+1}$ induces a bundle metric $g^{\mathcal{S}(M)}(s_1, s_2) :=< v_1, v_2 >$ for $s_i = [\tilde{H}, v_i] \in \mathcal{S}(M)$, $i = 1, 2$, which is parallel with respect to $\nabla^{\mathcal{S}(M)}$. Choosing a representative $g \in c$ turns the spin structure $(\mathcal{Q}^g, f^g)$ into a reduction of the conformal spin structure $\mathcal{Q}^0$ and its first prolongation $\mathcal{Q}^1$. Thus, we have the following isomorphisms:

$$\begin{aligned} \mathcal{S}(M) &\simeq \mathcal{Q}^g \times_{(Spin_0(p,q),\tilde{\rho})} \Delta_{p+1,q+1}, \\ TM &\simeq \mathcal{Q}^g \times_{(Spin_0(p,q),Ad\circ\lambda)} \mathfrak{b}_{-1}, \\ T^*M &\simeq \mathcal{Q}^g \times_{(Spin_0(p,q),Ad\circ\lambda)} \mathfrak{b}_1, \\ \mathfrak{so}(M, g) &\simeq \mathcal{Q}^g \times_{(Spin_0(p,q),Ad\circ\lambda)} \mathfrak{so}(p, q). \end{aligned}$$

They yield actions of $V = TM, T^*M, \mathfrak{so}(TM, g)$ on $\mathcal{S}(M)$ by

$$\tilde{\rho}^g(\Theta)s := [q, \tilde{\rho}_* \circ \lambda_*^{-1}([q]^{-1}(\Theta))v],$$

where $s = [q, v] \in \mathcal{Q}^g \times_{(Spin_0(p,q),\tilde{\rho})} \Delta_{p+1,q+1}$, $\Theta \in V$, and $[q] : W \to V$ is the isomorphism $[q](w) := [q, w]$ for $W = \mathfrak{b}_{-1}, \mathfrak{b}_1, \mathfrak{so}(p, q)$. Thus we can formulate:

Lemma 5.6 *The covariant derivative $\nabla^{\mathcal{S}(M)}$ induced by the normal conformal spin connection is given with respect to a metric $g \in c$ by*

$$\nabla_X^{\mathcal{S}(M)} s = \nabla_X^g s + \tilde{\rho}^g(X)s - \tilde{\rho}^g(P^g(X))s,$$

where $s \in \Gamma(\mathcal{S}(M))$, $X \in \mathfrak{X}(M)$, $P^g \in \Omega^1(M, T^*M)$ and ∇^g is the associated covariant derivative induced by the spin connection of (M, g).

Proof. Recall the maps $\tilde{\sigma}^{A^g} : \mathcal{Q}^g \to \mathcal{Q}^1(M)$ induced by $\tilde{\gamma}^g$. Representing s locally as $s = [q, v] = [\tilde{\sigma}^{A^g}(q), v]$ for a local section $q : U \to \mathcal{Q}^g$ and a smooth map $v : U \to \Delta_{p+1,q+1}$, it follows from equation (3.13) that

$$\begin{aligned}
\nabla_X^{\mathcal{S}(M)} s =& [q, X(v) + \tilde{\rho}_*\Big(\tilde{w}^{nc}(d(\tilde{\sigma}^{A^g} \circ q)(X))\Big)v] \\
=& [q, X(v) + \tilde{\rho}_*\Big(\lambda_*^{-1}(\Theta^{\mathcal{P}^0}_{f(q)}(d(f \circ q)_x(X)) + \tilde{A}^g_q(d(q)_x(X))\Big)v] \\
&- [q, \tilde{\rho}_* \circ \lambda_*^{-1}\Big(\sum_i P^g(d\pi(d(f \circ q)_x(X)), s_i)e_i^*)\Big)v] \\
=& \nabla_X^g \psi + \tilde{\rho}^g(X)\psi - \tilde{\rho}^g(P^g(X))\psi,
\end{aligned}$$

which proves the lemma. $\square$

Remark 5.7 Until now, one could have also formulated everything in terms of an arbitrary finite dimensional representation of the spin group $Spin_0(p+1, q+1)$. But, in what will follow, we will need special properties of the spin representation.

Now we are going to identify the standard spin tractor bundle with a direct sum of two copies of $S(M, g)$ for a fixed $g \in c$.

Lemma 5.8 *For any $g \in c$, there exists a vector bundle isomorphism*

$$\Psi^g : \mathcal{S}(M) \to S(M, g) \oplus S(M, g) =: \mathcal{S}(M)_g.$$

Proof. Let $s \in \mathcal{S}(M)$ be a spin tractor. There are $q \in \mathcal{Q}^g$ and $v \in \Delta_{p+1,q+1}$ such that $s = [q, v] = [\tilde{\sigma}^{A^g}(q, v]$, locally. Remember the isotropic basis $\{f_0, e_1, \ldots, e_n, f_{n+1}\}$

on $(\mathbb{R}^{n+2}, \langle\cdot,\cdot\rangle_{p+1,q+1})$ and the action of Clifford multiplication of $\mathbb{R}^{n+2}$ on the representation space $\Delta_{p+1,q+1}$. We define two subspaces in $\Delta_{p+1,q+1}$ by

$$\begin{aligned} W^+ :=&\{v \in \Delta_{p+1,q+1} \mid f_{n+1}\cdot v = 0\},\\ W^- :=&\{v \in \Delta_{p+1,q+1} \mid f_0 \cdot v = 0\}. \end{aligned}$$

These subspaces are invariant under the action of $Spin_0(p,q)$, since f_0 and f_{n+1} commute with elements of $Spin_0(p,q)$. The representation $\tilde{\rho}$ restricted to $Spin_0(p,q)$ decomposes in two representations

$$\tilde{\rho}_{|Spin_0(p,q)} = \tilde{\rho}^+ \oplus \tilde{\rho}^- : Spin_0(p,q) \to GL(W^+) \oplus GL(W^-).$$

These two representations $\tilde{\rho}^\pm$ are both equivalent to the spin representation $\kappa_{p,q}$: $Spin_0(p,q) \to GL(\Delta_{p,q})$. From the vector space isomorphism $W^+ \oplus W^+ \ni (w_1,w_2) \mapsto w_1 + f_0\cdot w_2 \in \Delta_{p+1,q+1}$ we define

$$\begin{aligned} \Psi^g : \mathcal{S}(M) &\to S(M,g)\oplus S(M,g)\\ s &\mapsto \Psi^g(s) := ([q,w_1],[q,w_2]), \end{aligned}$$

where $s = [q,v] \in \mathcal{Q}^g \times_{(Spin_0(p,q),\tilde{\rho})} \Delta_{p+1,q+1}$ and $v = w_1 + f_0\cdot w_2$ for $w_1, w_2 \in W^+$. By construction Ψ^g is a vector bundle isomorphism. $\square$

We will call Ψ^g the $g-$trivialization of the standard spin tractor bundle. Again, we may define the $g-$metric representation of $\nabla^{\mathcal{S}(M)}$ by $\nabla^{g,\mathcal{S}(M)}_X(s_g) := \Psi^g \circ \nabla^{\mathcal{S}(M)}_X((\Psi^g)^{-1}s_g)$ for $s_g \in \mathcal{S}(M)_g$. Then we have:

Lemma 5.9 *In terms of the metric $g \in c$ one has*

$$\nabla^{g,\mathcal{S}(M)}_X \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix} \nabla^{S(M,g)}_X & X\cdot \\ \frac{1}{2}P^g(X)^\natural\cdot & \nabla^{S(M,g)}_X \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

where $\psi_i \in \Gamma(S(M,g))$, $P^g \in \Omega^1(M,T^*M)$ is the Schouten tensor with respect to g, and $\nabla^{S(M,g)}$ is the covariant derivative on $S(M,g)$ induced by the spin connection form $\tilde{A}^g$. The spin tractor curvature in terms of g is

$$\mathcal{R}^{g,\mathcal{S}(M)}(X,Y)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \frac{1}{2}\begin{pmatrix} W^g(X,Y)\cdot & \\ C^g(X,Y,\cdot)^\natural\cdot & W^g(X,Y)\cdot \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

where the Weyl tensor $W^g(X,Y)$ acts as a $2-$form on spinors by Clifford multiplication, and C^g is the Cotton tensor.

Proof. The proof of the first equation is based on the actions $\tilde{\rho}^g$ on the spin tractor bundle with respect to the decomposition $\mathcal{S}(M)_g \simeq S(M,g)\oplus S(M,g)$. In view of Lemma 5.6, we have to consider the actions $\tilde{\rho}^g(X)$ and $\tilde{\rho}^g(P(X))$ on $(\psi_1 + f_0\cdot\psi_2) \in \mathcal{S}(M)$ and to use Lemma 3.10, in order to obtain

$$\begin{aligned}\tilde{\rho}^g(X)(\psi_1 + f_0 \cdot \psi_2) =&(-\frac{1}{2}X \cdot f_{n+1}\cdot)(\psi_1 + f_0 \cdot \psi_2) = X \cdot \psi_2,\\ \tilde{\rho}^g(P^g(X))(\psi_1 + f_0 \cdot \psi_2) =&(\frac{1}{2}P^g(X)^\natural \cdot f_0)(\psi_1 + f_0 \cdot \psi_2) = -\frac{1}{2}f_0 \cdot P^g(X)^\natural \cdot \psi_1.\end{aligned}$$

where $X, P^g(X)^\natural \in TM$. Finally, again by Lemma 3.10, it holds that

$$\nabla^g_X(\psi_1 + f_0 \cdot \psi_2) = \nabla^{S(M,g)}_X \psi_1 + f_0 \cdot \nabla^{S(M,g)}_X \psi_2.$$

All together yields the desired matrix representation of $\nabla^{g,\mathcal{S}(M)}$.

Let $X, Y \in \mathfrak{X}(M)$. The curvature formula is computed to be

$$\mathcal{R}^{g,\mathcal{S}(M)}(X,Y) = \nabla^{g,\mathcal{S}(M)}_X \nabla^{g,\mathcal{S}(M)}_Y - \nabla^{g,\mathcal{S}(M)}_Y \nabla^{g,\mathcal{S}(M)}_X - \nabla^{g,\mathcal{S}(M)}_{[X,Y]} = \begin{pmatrix} R_1 & 0 \\ R_3 & R_4 \end{pmatrix},$$

with

$$\begin{aligned}R_1 :=&\mathcal{R}^{S(M,g)}(X,Y) + \frac{1}{2}(X \cdot P^g(Y)^\natural \cdot - Y \cdot P^g(X)^\natural \cdot),\\ R_3 :=&\frac{1}{2}(\nabla_X(P^g(Y)^\natural) \cdot - \nabla_Y(P^g(X)^\natural) \cdot - P^g([X,Y])^\natural \cdot),\\ R_4 :=&\mathcal{R}^{S(M,g)}(X,Y) + \frac{1}{2}(P^g(X)^\natural \cdot Y \cdot - P^g(Y)^\natural \cdot X \cdot),\end{aligned}$$

and the observations

$$\begin{aligned}\mathcal{R}^{S(M,g)}(X,Y) + \frac{1}{2}(X \cdot P^g(Y)^\natural \cdot - Y \cdot P^g(X)^\natural \cdot) =&\frac{1}{2}W^g(X,Y)\cdot\\ \frac{1}{2}(\nabla_X(P^g(Y)^\natural) \cdot - \nabla_Y(P^g(X)^\natural) \cdot - P^g([X,Y])^\natural \cdot) =&\frac{1}{2}C^g(X,Y,\cdot)^\natural \cdot .\end{aligned}$$

Note that the curvature is acting in a normal way on a spinor, whereas the Weyl curvature acts by Clifford multiplication. □

It remains to identify $\mathcal{S}(M)_g$ with respect to different metrics from the conformal class.

Lemma 5.10 *Let $g, \hat{g} = e^{2\sigma}g \in c$. Then, the map $T^{\mathcal{S}(M)}(g,\sigma) := \Psi^{\hat{g}} \circ (\Psi^g)^{-1}$ is given by the matrix*

$$T^{\mathcal{S}(M)}(g,\sigma) = F_\sigma \oplus F_\sigma \begin{pmatrix} e^{\frac{1}{2}\sigma} & 0 \\ \frac{1}{2}e^{-\frac{1}{2}\sigma}\operatorname{grad}^g(\sigma)\cdot & e^{-\frac{1}{2}\sigma} \end{pmatrix}, \tag{5.2}$$

where $F_\sigma = \hat{\cdot} : S(M,g) \to S(M,\hat{g})$ is the canonical vector bundle isomorphism for conformally related metrics.

Proof. The proof runs through the same lines as the proof of Lemma 5.5. A choice of $g, \hat{g} = e^{2\sigma} g \in c$ implies that $\mathcal{Q}^g$ and $\mathcal{Q}^{\hat{g}}$ are $Spin_0(p,q) \to CSpin_0(p,q) \simeq \tilde{B}_0-$reductions of the conformal spin structure $(\mathcal{Q}^0, f^0)$, hence

$$\mathcal{Q}^g \times_{Spin_0(p,q)} \tilde{B}_0 \simeq \mathcal{Q}^0 \simeq \mathcal{Q}^{\hat{g}} \times_{Spin_0(p,q)} \tilde{B}_0. \tag{5.3}$$

Let $\gamma : I \to U \subset M$ be a smooth curve, and $q : U \to \mathcal{Q}^g$, $\hat{q} : u \to \mathcal{Q}^{\hat{g}}$ be local sections. A tangent vector $\xi \in T_{\tilde{q}}\mathcal{Q}^0$ for $\tilde{q} \in \mathcal{Q}^0$ can be represented as $\xi = \dot{\tilde{\delta}}(0)$, where $\tilde{\delta} : I \to \mathcal{Q}^0$ is a smooth curve such that $\tilde{\delta}(0) = \tilde{q}$. The isomorphisms (5.3) lead to representations $\tilde{\delta}(t) = [q(t), \tilde{b}(t)] \in \mathcal{Q}^g \times_{Spin_0(p,q)} \tilde{B}_0$ and $\tilde{\delta}(t) = [\hat{q}(t), \hat{\tilde{b}}(0)] \in \mathcal{Q}^{\hat{g}} \times_{Spin_0(p,q)} \tilde{B}_0$, where we have defined $q(t) := q \circ \gamma(t)$ and $\hat{q}(t) := \hat{q} \circ \gamma(t)$. The local sections $q, \hat{q}$ are also local sections of $\mathcal{Q}^0$, hence $\hat{q}(t) = q(t) \cdot \tilde{a}(t)$ for a curve $\tilde{a} : I \to \tilde{B}_0$. It follows that $\hat{\tilde{b}}(t) = \tilde{a}^{-1}(t) \cdot \tilde{b}(t)$. In this setting, the difference of the connections $\tilde{\gamma}^g$ and $\tilde{\gamma}^{\hat{g}}$ is computed, similar to the standard tractor case, to be

$$\tilde{\gamma}^{\hat{g}}_{\tilde{q}}(\xi) - \tilde{\gamma}^{g}_{\tilde{q}}(\xi) = \lambda_*^{-1} \left([Z, \theta^{\mathcal{P}^0}_{f^0(\tilde{q})}(df^0_{\tilde{q}}(\xi))] \right)$$

for $Z := M(0,(0,0),[f^0(\tilde{q})]^{-1} d\sigma) \in \mathfrak{b}_1$ and $\theta^{\mathcal{P}^0}_{f^0(\tilde{q})}(df^0_{\tilde{q}}(\xi)) = M(\dot{\gamma}(0),(0,0),0) \in \mathfrak{b}_{-1}$. Thus it follows, that the reduction maps are related by

$$\tilde{\sigma}^{A^g}(q) = \tilde{\sigma}^{A^{\hat{g}}}(\hat{q}) \cdot \tilde{a}^{-1}(0) \cdot \lambda_*^{-1}(exp(-Z)) \cdot \tilde{a}(0) = \tilde{\sigma}^{A^{\hat{g}}}(\hat{q}) \cdot \tilde{b},$$

where $\tilde{b} := \tilde{b}_0^{-1} \cdot \tilde{b}_1$ for factors mentioned in Remark 3.11. Comparing the trivializations Ψ^g and $\Psi^{\hat{g}}$, which are essentially given by the reduction maps, proves the claim of the lemma by

$$\begin{aligned}
s_{|U} =& [\tilde{\sigma}^{A^g}(q), v_1 + f_0 \cdot v_2] \overset{g}{\mapsto} [q, v_1] + [q, v_2] = (\psi_1, \psi_2) \\
s_{|U} =& [\tilde{\sigma}^{A^g}(q), v_1 + f_0 \cdot v_2] = [\tilde{\sigma}^{A^{\hat{g}}}(\hat{q}), \tilde{b} \cdot (v_1 + f_0 \cdot v_2)] \\
=& [\tilde{\sigma}^{A^{\hat{g}}}(\hat{q}), e^{\frac{1}{2}\sigma} v_1 + e^{-\frac{1}{2}\sigma} f_0 \cdot (-\frac{1}{2} Z \cdot v_1 + v_2)] \\
& \overset{\hat{g}}{\mapsto} (e^{\frac{1}{2}\sigma} F_\sigma(\psi_1), e^{-\frac{1}{2}\sigma} F_\sigma(\frac{1}{2}\, \mathrm{grad}^g(\sigma) \cdot \psi_1 + \psi_2))
\end{aligned}$$

for $s \in \Gamma\left(\mathcal{S}(M)\right)$ and $F_\sigma : S(M,g) \to S(M,\hat{g})$. $\square$

Remark 5.11 We can derive some conformal transformation laws from the last two lemmas: Let $g, \hat{g} = e^{2\sigma} g \in c$. Firstly, since the spin tractor curvature $\mathcal{R}^{\mathcal{S}(M)}$ represented in conformally equivalent metrics satisfies $\mathcal{R}^{\hat{g},\mathcal{S}(M)} \circ T^{\mathcal{S}(M)}(g,\sigma) = T^{\mathcal{S}(M)}(g,\sigma) \circ \mathcal{R}^{g,\mathcal{S}(M)}$, we have for $X, Y, Z \in \mathfrak{X}(M)$ and $\psi \in \Gamma\big(S(M,g)\big)$ that

$$W^{\hat{g}}(X,Y) \hat{\cdot} \hat{\psi} = F_\sigma\big(W^g(X,Y) \cdot \psi\big),$$

which represents the known fact that the Weyl tensor is conformally covariant in the sense of $W^{\hat{g}} = e^{-2\sigma}W^g$, considered as a $(0,4)-$tensor.

Secondly, recalling $\hat{X} = e^{-\sigma}X$, by definition, and that $\cdot^{\natural}$ is metric dependend, we obtain the following relation,

$$\begin{aligned}
e^{\frac{1}{2}\sigma}C^{\hat{g}}(X,Y)^{\hat{\natural}}\hat{\cdot}\hat{\psi} + \frac{1}{2}e^{-\frac{1}{2}\sigma}W^{\hat{g}}(X,Y)\hat{\cdot}F_\sigma(\mathrm{grad}^g(\sigma)\cdot\psi) \\
=&\frac{1}{2}e^{-\frac{1}{2}\sigma}F_\sigma\big(\,\mathrm{grad}^g(\sigma)\cdot W^g(X,Y)\cdot\psi\big) + e^{-\frac{1}{2}\sigma}F_\sigma\big(C^g(X,Y)^{\natural}\cdot\psi\big),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
e^{\sigma}C^{\hat{g}}(X,Y)^{\hat{\natural}}\hat{\cdot}\hat{\psi} =& F_\sigma\big(C^g(X,Y)^{\natural}\cdot\psi\big) \\
&+ \frac{1}{2}F_\sigma\big(\,\mathrm{grad}^g(\sigma)\cdot W^g(X,Y)\cdot\psi - W^g(X,Y)\cdot\mathrm{grad}^g(\sigma)\cdot\psi\big) \\
=& F_\sigma\big(C^g(X,Y)^{\natural}\cdot\psi\big) - F_\sigma\big(W^g(X,Y,\mathrm{grad}^g(\sigma),\cdot)^{\natural}\cdot\psi\big).
\end{aligned}$$

This recovers the conformal transformation law of the Cotton tensor, i.e., $C^{\hat{g}}(X,Y,Z) = C^g(X,Y,Z) - W^g(X,Y,\mathrm{grad}^g(\sigma),Z)$ for all $X,Y,Z \in \mathfrak{X}(M)$.

From now on we will omit the index g attached to curvature tensors arising from the metric g, as long as it is clear from the context which representative we are working with.

5.3 Operators on tractor bundles

This section deals with the typical operators acting on our tractor bundles. Their formulations involve tensor products between standard and standard spin tractor bundles. On tensor products, we will always work with the tensor product covariant derivative. The exterior differential and co-differential acting on $k-$form with values in $\mathcal{S}(M)$ were given by

$$\begin{aligned}
d^{\nabla^{\mathcal{S}(M)}}w^k(X_0,\dots,X_k) &= \sum_{i=0}^{k}(-1)^i(\nabla^{\Lambda^kM\otimes\mathcal{S}(M)}_{X_i}w^k)(X_0,\dots,\hat{X}_i,\dots,X_k) \\
\delta^{\nabla^{\mathcal{S}(M)}}w^k(X_1,\dots,X_{k-1}) &= -\sum_{i=1}^{n}\varepsilon_i(\nabla^{\Lambda^kM\otimes\mathcal{S}(M)}_{s_i}w^k)(s_i,X_1,\dots,X_{k-1}),\ k\geq 1,
\end{aligned}$$

where $w^k \in \Omega^k(M,\mathcal{S}(M)) = \Gamma(\Lambda^kM\otimes\mathcal{S}(M))$, $s=\{s_i\}: U \to \mathcal{P}^g$ is a local section and $X_i \in \mathfrak{X}(M)$. Note that $\delta^{\nabla^{\mathcal{S}(M)}}$ depends on the metric g and that $d^{\nabla^{\mathcal{S}(M)}}\circ d^{\nabla^{\mathcal{S}(M)}} = \mathcal{R}^{\mathcal{S}(M)}$ does not vanish in general. The two operators above yield together two differential operators

$$\begin{aligned}
\Delta_k =& \delta^{\nabla^{\mathcal{S}(M)}}\circ d^{\nabla^{\mathcal{S}(M)}} + d^{\nabla^{\mathcal{S}(M)}}\circ\delta^{\nabla^{\mathcal{S}(M)}}, \\
\Delta_g^{\Lambda^kM\otimes\mathcal{S}(M)} =& Tr_g(\nabla^{T^*M\otimes\Lambda^kM\otimes\mathcal{S}(M)}\circ\nabla^{\Lambda^kM\otimes\mathcal{S}(M)}),
\end{aligned}$$

the Hodge-Laplacian and the Bochner-Laplacian of the standard spin tractor bundle. Both depend on the metric and act on $k-$forms with values in the standard spin tractor bundle. The Weitzenböck formula for $1-$forms $\eta \in \Omega^1(M, \mathcal{S}(M))$ reads

$$(\Delta_g^{T^*M\otimes\mathcal{S}(M)}\eta)(V) = -(\Delta_1\eta)(V) - \sum_i \varepsilon_i(\mathcal{R}^{T^*M\otimes\mathcal{S}(M)}(V,s_i)\eta)(s_i),$$

where $V \in \mathfrak{X}(M)$ and $\{s_i\} : U \to \mathcal{P}^g$ is a local section, compare [Xin96][Proposition 1.3.4]. In case of exact $1-$forms $\eta = d^{\nabla^{\mathcal{S}(M)}}s = \nabla^{\mathcal{S}(M)}s$ for some $s \in \Gamma(\mathcal{S}(M))$, this gives

$$\begin{aligned}&(\Delta_g^{T^*M\otimes\mathcal{S}(M)}d^{\nabla^{\mathcal{S}(M)}}s)(V) - (d^{\nabla^{\mathcal{S}(M)}}\Delta_g^{\mathcal{S}(M)}s)(V)\\ &= -\delta^{\nabla^{\mathcal{S}(M)}}(\mathcal{R}^{\mathcal{S}(M)}s)(V) - \sum_i \varepsilon_i\mathcal{R}^{\mathcal{S}(M)}(V,s_i)(d^{\nabla^{\mathcal{S}(M)}}s(s_i)) + d^{\nabla^{\mathcal{S}(M)}}s(Ric(V)^\natural).\end{aligned} \tag{5.4}$$

Since our $g-$trivializations of the standard tractor bundle involve the tangent bundle and not its dual, it will be useful to note

$$\Delta_g^{TM\otimes\mathcal{S}(M)}((d^{\nabla^{\mathcal{S}(M)}}s)^\natural) = (\Delta_g^{T^*M\otimes\mathcal{S}(M)}(d^{\nabla^{\mathcal{S}(M)}}s))^\natural, \tag{5.5}$$

and to consider the divergence operator $\mathrm{div}^\otimes : \Gamma(TM\otimes\mathcal{S}(M)) \to \Gamma(\mathcal{S}(M))$, defined by

$$\begin{aligned}\mathrm{div}^\otimes(Y\otimes s) &:= -\,\delta^{\nabla^{\mathcal{S}(M)}}\left((Y\otimes s)^\flat\right) = \mathrm{div}(Y)\otimes s + \nabla_Y^{\mathcal{S}(M)}s\\ &\overset{\text{loc.}}{=} \sum_i \varepsilon_i\sigma^i(\nabla_{s_i}^{TM\otimes\mathcal{S}(M)}(Y\otimes s))\end{aligned}$$

for all $Y\otimes s \in \Gamma(TM\otimes\mathcal{S}(M))$. This definition extends to arbitrary vector bundles with a covariant derivative. Note that $div^\otimes$ is metric dependent. For future reference, let us state the $g-$metric representation of the divergence of the spin tractor curvature.

Lemma 5.12 *Let $g \in c$, $V \in \mathfrak{X}(M)$ and $s_g \in \Gamma(\mathcal{S}(M)_g)$. Then one has*

$$(\delta^{\nabla^{g,\mathcal{S}(M)}}\mathcal{R}^{g,\mathcal{S}(M)})s_g(V) = \frac{1}{2}\begin{pmatrix}(n-4)C(V)\cdot & 0\\ -B(V)^\natural\cdot & (n-4)C(V)\cdot\end{pmatrix}s_g,$$

where the Cotton tensor C is considered as a $1-$form with values in $2-$forms, and B is the Bach tensor, both with respect to g.

Proof. Consider a local section $\{s_i\} : U \to \mathcal{P}^g$, which is synchronous at $x \in U$, see Remark 2.2. Then at x, the following holds,

$$(\delta^{\nabla^{g,\mathcal{S}(M)}}\mathcal{R}^{g,\mathcal{S}(M)})s_g(V) = -\sum_i \varepsilon_i(\nabla_{s_i}^{g,\mathcal{S}(M)}\mathcal{R}^{g,\mathcal{S}(M)})s_g(s_i,V)$$

$$
\begin{aligned}
&= -\sum_i \varepsilon_i (\nabla^{g,\mathcal{S}(M)}_{s_i}(\mathcal{R}^{g,\mathcal{S}(M)} s_g))(s_i, V) \\
&\quad + \sum_i \varepsilon_i \mathcal{R}^{g,\mathcal{S}(M)}(s_i, V)\nabla^{g,\mathcal{S}(M)}_{s_i} s.
\end{aligned}
$$

Computing the summands seperately, we get

$$
\begin{aligned}
&\sum_i \varepsilon_i (\nabla^{g,\mathcal{S}(M)}_{s_i}(\mathcal{R}^{g,\mathcal{S}(M)} s_g))(s_i, V) \\
=&\frac{1}{2}\sum_i \varepsilon_i \begin{pmatrix} \nabla^{S(M,g)}_{s_i}(W(\cdot,\cdot)\cdot) + s_i \cdot C(s_i,V)^\natural\cdot & s_i \cdot W(s_i,V)\cdot \\ \frac{1}{2}P(s_i)^\natural \cdot W(s_i,V)\cdot + \nabla^{S(M,g)}_{s_i}(C(\cdot,\cdot)^\natural\cdot)(s_i,V) & \nabla^{S(M,g)}_{s_i}(W(\cdot,\cdot)\cdot)(s_i,V) \end{pmatrix} s_g,
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\sum_i \varepsilon_i \mathcal{R}^{g,\mathcal{S}(M)}(s_i, V)(\nabla^{g,\mathcal{S}(M)}_{s_i} s_g) \\
=&\frac{1}{2}\sum_i \varepsilon_i \begin{pmatrix} W(s_i,V)\cdot\nabla^{S(M,g)}_{s_i} & W(s_i,V)\cdot s_i\cdot \\ C(s_i,V)^\natural\cdot\nabla^{S(M,g)}_{s_i} + \frac{1}{2}W(s_i,V)\cdot P(s_i)^\natural\cdot & C(s_i,V)^\natural\cdot + W(s_i,V)\nabla^{S(M,g)}_{s_i} \end{pmatrix} s_g.
\end{aligned}
$$

Finally, from $-\delta(W)(X,Y,Z) = (n-3)C(Z,Y,X)$, Lemma 2.6 and the simple identities $\sum_i \varepsilon_i s_i \cdot W(s_i,V)\cdot = 0 = \sum_i \varepsilon_i W(s_i,V)\cdot s_i\cdot$, we get

$$
\begin{aligned}
&(\delta^{\nabla^{g,\mathcal{S}(M)}}\mathcal{R}^{g,\mathcal{S}(M)})s_g(V) \\
=&\frac{1}{2}\begin{pmatrix} (n-4)C(V)\cdot & 0 \\ -\sum_i \varepsilon_i(\nabla_{s_i}C)(s_i,V)^\natural\cdot - \sum_{k,l}\varepsilon_k\varepsilon_l W(P(s_l)^\natural, V, s_k, s_l)s_k\cdot & (n-4)C(V)\cdot \end{pmatrix} s_g.
\end{aligned}
$$

Up to a sign the left lower entry in the matrix is the Bach tensor, which proves the lemma. □

Remark 5.13 All the tensor products of tractor bundles can be decomposed into a tensor product of $g-$trivializations for any $g \in c$. We can also $g-$trivialize certain parts of the tensor product. Let us demonstrate how this works for $\mathcal{S}^{\mathcal{T}}(M) := \mathcal{T}(M)\otimes\mathcal{S}(M)$: On $\mathcal{S}^{\mathcal{T}}(M)$ we have the induced covariant derivative

$$\nabla^{\otimes} := \nabla^{\mathcal{T}(M)} \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^{\mathcal{S}(M)}.$$

If $g \in c$, then we have a vector bundle isomorphism

$$
\begin{aligned}
\Theta^g : \mathcal{S}^{\mathcal{T}}(M) &\to \mathcal{T}(M)_g \otimes \mathcal{S}(M) =: \mathcal{S}^{\mathcal{T}}(M)_g \\
t\otimes s &\mapsto \Theta^g(t\otimes s) := \Phi^g(t)\otimes s = t_g \otimes s,
\end{aligned}
$$

induced by the $g-$trivialization Φ^g. By $T^{\otimes}(g,\sigma) := \Theta^{\hat{g}} \circ (\Theta^g)^{-1}$ we denote the vector bundle isomorphism from $\mathcal{S}^{\mathcal{T}}(M)_g$ to $\mathcal{S}^{\mathcal{T}}(M)_{\hat{g}}$. From the direct sum structure of $\mathcal{T}(M)_g$

we get

$$\begin{aligned}\mathcal{S}^{\mathcal{T}}(M)_g &= \mathcal{S}(M) \oplus [TM \otimes \mathcal{S}(M)] \oplus \mathcal{S}(M)\\ t_g \otimes s &= (s_1, \eta, s_2).\end{aligned}$$

Finally, let us note that we could additionally $g-$trivialize the standard spin tractor parts, which would yield a complete $g-$trivialization of the bundle $\mathcal{S}^{\mathcal{T}}(M)$.

In what will follow, we are going to made ad-hoc definitions of certain operators and to prove properties of them. These operators are special candidates of splitting-operators on tractor bundles and their formal adjoints, compare [ČSS01, Šil06, Ham09] for general splitting-operators, and [ČG01, ČG00] for tractor D-operators on tractor bundles.

Let us define the tractor D-operator of the spinor bundle $S(M,g)$ by

$$\begin{aligned}D^{S(M,g)}(g,\eta) &: \Gamma\big(S(M,g)\big) \to \Gamma\big(\mathcal{S}(M)_g\big)\\ &\psi \mapsto D^{S(M,g)}(g,\eta)\psi := \begin{pmatrix}(\eta+\frac{n-1}{2})\psi\\ \frac{1}{2}\not{D}\psi\end{pmatrix},\end{aligned}$$

for $\eta \in \mathbb{R}$. Here, $\not{D}$ denotes the Dirac operator of the spin manifold (M,g). This operator satisfies a conformal covariance of bi-degree $(\eta, \eta - \frac{1}{2})$:

Lemma 5.14 *Let $\hat{g} = e^{2\sigma} g$ be conformally equivalent to g. Then one has*

$$D^{S(M,\hat{g})}(\hat{g},\eta)(e^{\eta\sigma}\hat{\psi}) = e^{(\eta-\frac{1}{2})\sigma} T^{\mathcal{S}(M)}(g,\sigma) D^{S(M,g)}(g,\eta)\psi$$

for any $\psi \in \Gamma\left(S(M,g)\right)$ and $\hat{\psi} = F_\sigma(\psi) \in \Gamma\left(S(M,g)\right)$, i.e., $D^{S(M,g)}(g,\eta)$ is a conformally covariant differential operator of bi-degree $(\eta, \eta-\frac{1}{2})$.

Proof. The proof is based on calculations involving the conformal covariance of the Dirac operator, i.e., $e^{\frac{n+1}{2}\sigma}\hat{\not{D}}(e^{\frac{1-n}{2}\sigma}\hat{\psi}) = \widehat{\not{D}\psi}(= F_\sigma(\not{D}\psi))$, and the product rule $\not{D}(f\psi) = \mathrm{grad}^g(f)\cdot\psi + f\not{D}\psi$ for a function $f \in \mathcal{C}^\infty(M)$. Let us start to compute

$$\begin{aligned}D^{S(M,\hat{g})}(\hat{g},\eta)(e^{\eta\sigma}\hat{\psi}) &= \begin{pmatrix}(\eta+\frac{n}{2}-\frac{1}{2})e^{\eta\sigma}\hat{\psi}\\ \frac{1}{2}\hat{\not{D}}(e^{\eta\sigma}\hat{\psi})\end{pmatrix} = \begin{pmatrix}(\eta+\frac{n}{2}-\frac{1}{2})e^{\eta\sigma}\hat{\psi}\\ \frac{1}{2}\hat{\not{D}}(e^{\frac{1-n}{2}\sigma}e^{\frac{n-1}{2}\sigma}e^{\eta\sigma})\hat{\psi}\end{pmatrix}\\ &= e^{(\eta-\frac{1}{2})\sigma}\begin{pmatrix}(\eta+\frac{n}{2}-\frac{1}{2})e^{\frac{1}{2}\sigma}\hat{\psi}\\ \frac{1}{2}e^{(\frac{1}{2}-\eta)\sigma}e^{-\frac{n+1}{2}\sigma}F_\sigma\big(\not{D}(e^{(\eta+\frac{n}{2}-\frac{1}{2})\sigma}\psi)\big)\end{pmatrix}\\ &= e^{(\eta-\frac{1}{2})\sigma}\begin{pmatrix}(\eta+\frac{n}{2}-\frac{1}{2})e^{\frac{1}{2}\sigma}\hat{\psi}\\ \frac{1}{2}e^{-\frac{1}{2}\sigma}\big[(\frac{n}{2}+\eta-\frac{1}{2})F_\sigma(\mathrm{grad}^g(\sigma)\cdot\psi) + F_\sigma(\not{D}\psi)\big]\end{pmatrix}.\end{aligned}$$

On the other hand we have

$$T^{\mathcal{S}(M)}(g,\sigma)D^{S(M,g)}(g,\eta)\psi = F_\sigma \oplus F_\sigma \begin{pmatrix}e^{\frac{1}{2}\sigma} & 0\\ \frac{1}{2}e^{-\frac{1}{2}\sigma}\,\mathrm{grad}^g(\sigma) & e^{-\frac{1}{2}\sigma}\end{pmatrix}\begin{pmatrix}(\eta+\frac{n}{2}-\frac{1}{2})\psi\\ \frac{1}{2}\not{D}\psi\end{pmatrix}$$

$$
\begin{aligned}
&= F_\sigma \oplus F_\sigma \begin{pmatrix} (\eta+\frac{n}{2}-\frac{1}{2})e^{\frac{1}{2}\sigma}\psi \\ \frac{1}{2}(\eta+\frac{n}{2}-\frac{1}{2})e^{-\frac{1}{2}\sigma}\operatorname{grad}^g(\sigma)\cdot\psi+\frac{1}{2}e^{-\frac{1}{2}\sigma}\not{D}\psi \end{pmatrix} \\
&= \begin{pmatrix} e^{\frac{1}{2}\sigma}(\eta+\frac{n}{2}-\frac{1}{2})\hat{\psi} \\ \frac{1}{2}(\eta+\frac{n}{2}-\frac{1}{2})e^{-\frac{1}{2}\sigma}F_\sigma(\operatorname{grad}^g(\sigma)\cdot\psi)+\frac{1}{2}e^{-\frac{1}{2}\sigma}F_\sigma(\not{D}\psi) \end{pmatrix}.
\end{aligned}
$$

This shows the conformal covariance of the tractor D-operator of the spinor bundle. □

Another operator is defined by

$$
\begin{aligned}
C^{S(M,g)}(g,\eta) &: \Gamma(\mathcal{S}(M)_g) \to \Gamma(S(M,g)) \\
s_g &= \begin{pmatrix}\psi\\ \phi\end{pmatrix} \mapsto C^{S(M,g)}(g,\eta)\begin{pmatrix}\psi\\ \phi\end{pmatrix} := \frac{1}{2}\not{D}\psi-(\eta+\frac{n}{2})\phi,
\end{aligned}
$$

for $\eta \in \mathbb{R}$. We call $C^{S(M,g)}(g,\eta)$ the tractor C-operator of the spinor bundle.

Analogously to $D^{S(M,g)}(g,\eta)$, we have:

Lemma 5.15 *Let $\hat{g} = e^{2\sigma}g$ be conformally related to g and let $s_g = \begin{pmatrix}\psi\\ \phi\end{pmatrix}$ be a $g-$trivialized spin tractor. Then one has*

$$
C^{S(M,\hat{g})}(\hat{g},\eta)(e^{\eta\sigma}T^{\mathcal{S}(M)}(g,\sigma)s_g) = e^{(\eta-\frac{1}{2})\sigma}F_\sigma(C^{S(M,g)}(g,\eta)s_g), \tag{5.6}
$$

where $F_\sigma : S(M,g) \to S(M,\hat{g})$ is the bundle isomorphism between conformally related spinor bundles. In particular, $C^{S(M,g)}(g,\eta)$ is a conformally covariant differential operator of bi-degree $(\eta, \eta-\frac{1}{2})$.

Proof. We compute

$$
\begin{aligned}
&C^{S(M,\hat{g})}(\hat{g},\eta)(e^{\eta\sigma}T^{\mathcal{S}(M)}(g,\sigma)s_g) \\
&\quad= C^{S(M,\hat{g})}(\hat{g},\eta)e^{\eta\sigma}\begin{pmatrix} e^{\frac{1}{2}}\hat{\psi} \\ \frac{1}{2}e^{-\frac{1}{2}\sigma}F_\sigma(\operatorname{grad}^g(\sigma)\cdot\psi)+e^{-\frac{1}{2}\sigma}\hat{\phi}\end{pmatrix} \\
&\quad= \frac{1}{2}\hat{\not{D}}(e^{(\eta+\frac{1}{2})\sigma}\hat{\psi})-(\eta+\frac{n}{2})e^{(\eta-\frac{1}{2})\sigma}(\frac{1}{2}F_\sigma(\operatorname{grad}^g(\sigma)\cdot\psi)+\hat{\phi}) \\
&\quad= \frac{1}{2}\hat{\not{D}}(e^{\frac{1-n}{2}\sigma}e^{(\eta+\frac{n}{2})\sigma}\hat{\psi})-(\eta+\frac{n}{2})e^{(\eta-\frac{1}{2})\sigma}(\frac{1}{2}F_\sigma(\operatorname{grad}^g(\sigma)\cdot\psi)+\hat{\phi}) \\
&\quad= \frac{1}{2}e^{-\frac{n+1}{2}\sigma}F_\sigma(\not{D}(e^{(\frac{n}{2}+\eta)\sigma}\psi))-(\eta+\frac{n}{2})e^{(\eta-\frac{1}{2})\sigma}(\frac{1}{2}F_\sigma(\operatorname{grad}^g(\sigma)\cdot\psi)+\hat{\phi}) \\
&\quad= \frac{1}{2}e^{(\eta-\frac{1}{2})\sigma}((\eta+\frac{n}{2})F_\sigma(\operatorname{grad}^g(\sigma)\cdot\psi)+F_\sigma(\not{D}\psi)) \\
&\qquad -(\eta+\frac{n}{2})e^{(\eta-\frac{1}{2})\sigma}(\frac{1}{2}F_\sigma(\operatorname{grad}^g(\sigma)\cdot\psi)+\hat{\phi}) \\
&\quad= e^{(\eta-\frac{1}{2})\sigma}[\frac{1}{2}F_\sigma(\not{D}\psi)-(\eta+\frac{n}{2})\hat{\phi}]
\end{aligned}
$$

$$=e^{(\eta-\frac{1}{2})\sigma}F_\sigma \circ C^{S(M,g)}(g,\eta)s_g,$$

which proves the lemma. □

Remark 5.16 Assume that M has empty boundary. The tractor C-operator of the spinor bundle is related to the formal adjoint of the tractor D-operator of the spinor bundle with respect to the L^2-scalar products induced by the bundle metrics $g^{\mathcal{S}(M)}$ of the standard spin tractor bundle and $<\cdot,\cdot>$ of the spinor bundle: First observe that

$$g^{\mathcal{S}(M)}\left(D^{S(M,g)}(g,\eta)\psi, \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right)_{L^2} =<\psi, \left(D^{S(M,g)}(g,\eta)\right)^* \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}>_{L^2}$$

for compactly supported sections $\psi,\varphi_1,\varphi_2 \in \Gamma_c\left(S(M,g)\right)$ and

$$\left(D^{S(M,g)}(g,\eta)\right)^* \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = (-1)^p i^s\left((-1)^{p+1}\frac{1}{2}\not{D}\varphi_1 - (\eta+\frac{n-1}{2})\varphi_2\right).$$

Thus we have that

$$C^{S(M,g)}(g,\eta)\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \propto \left(D^{S(M,g)}(g,\eta+\frac{1}{2})\right)^* \begin{pmatrix}(-1)^{p+1}\varphi_1\\ \varphi_2\end{pmatrix},$$

where $\propto$ denotes the proportional symbol, which yields the desired relation.

Corollary 5.17 *One has* $C^{S(M,g)}(g,\eta-\frac{1}{2})\circ D^{S(M,g)}(g,\eta)=0$ *for all* $\eta\in\mathbb{R}$.

Remark 5.18 The operators $D^{S(M,g)}(g,\eta)$ and $C^{S(M,g)}(g,\eta)$ have also interpretations on the tractor level. Namely, let us set

$$\begin{aligned} D_g^{S(M,g)}(\eta) : \Gamma\big(S(M,g)\big) &\to \Gamma\big(\mathcal{S}(M)\big)\\ \psi &\mapsto D_g^{S(M,g)}(\eta)\psi := (\Psi^g)^{-1}D^{S(M,g)}(g,\eta)\psi,\end{aligned}$$

as well as

$$\begin{aligned} C_g^{S(M,g)}(\eta) : \Gamma\big(S(M)\big) &\to \Gamma\big(S(M,g)\big)\\ s &\mapsto C_g^{S(M,g)}(\eta)s := C^{S(M,g)}(g,\eta)\circ\Psi^g(s).\end{aligned}$$

The results of Lemma 5.14 and Lemma 5.15 are equivalent to

$$\begin{aligned} D_{\hat g}^{S(M,\hat g)}(\eta)\big(e^{\eta\sigma}F_\sigma(\psi)\big) &= e^{(\eta-\frac{1}{2})\sigma}D_g^{S(M,g)}(\eta)\psi,\\ C_{\hat g}^{S(M,\hat g)}(\eta)(e^{\eta\sigma}s) &= e^{(\eta-\frac{1}{2})\sigma}F_\sigma\circ C_g^{S(M,g)}(\eta)s,\end{aligned}$$

where $\hat g = e^{2\sigma}g$, $\psi\in\Gamma\big(S(M,g)\big)$, $s\in\Gamma\big(\mathcal{S}(M)\big)$ and $\eta\in\mathbb{R}$.

Now the tensor product comes into play: The tractor D-operator of the standard spin tractor bundle is defined by

$$D^{\mathcal{S}(M)}(g,w) : \Gamma(\mathcal{S}(M)) \to \Gamma(\mathcal{S}^{\mathcal{T}(M)}(M)_g)$$
$$s \mapsto D^{\mathcal{S}(M)}(g,w)s := \begin{pmatrix} -\Box^{\mathcal{S}(M)}_{g,w} s \\ w_1(\nabla^{\mathcal{S}(M)} s)^{\natural} \\ ww_1 s \end{pmatrix},$$

where $w_1 := (n-2+2w)$, $\Box^{\mathcal{S}(M)}_{g,w} s := \Delta^{\mathcal{S}(M)}_g s + wJs$, J is the trace of the Schouten tensor P with respect to g and $(\nabla^{\mathcal{S}(M)} s)^{\natural} \overset{\text{loc.}}{=} \sum_{i=1}^n \varepsilon_i s_i \otimes \nabla^{\mathcal{S}(M)}_{s_i} s$. The action of $\cdot^{\natural}$ on a general $1-$form $\eta \in \Omega^1(M, \mathcal{S}(M))$ is defined, in a similar way, by $\Gamma(TM \otimes \mathcal{S}(M)) \ni \eta^{\natural} \overset{\text{loc.}}{:=} \sum_{i=1}^n \varepsilon_i s_i \otimes \eta(s_i)$. Again, an essential property of $D^{\mathcal{S}(M)}(g,w)$ is its transformation law under a conformal change of the metric:

Lemma 5.19 *Let g and $\hat{g} = e^{2\sigma} g$ be two conformally related metrics. Then for any $s \in \Gamma(\mathcal{S}(M))$ one has*

$$D^{\mathcal{S}(M)}(\hat{g}, w)(e^{w\sigma} s) = e^{(w-1)\sigma} T(g,\sigma) D^{\mathcal{S}(M)}(g,w)s,$$

i.e., $D^{\mathcal{S}(M)}(g,w)$ is a conformally covariant differential operator of bi-degree $(w, w-1)$.

Proof. Let us first recall some identities concerning the conformal transformations laws of involved objects: For any $\lambda \in \mathbb{R}$ one has

$$\begin{aligned} \Delta^{\mathcal{S}(M)}_g(e^{\lambda\sigma} s) =& e^{\lambda\sigma}(\Delta^{\mathcal{S}(M)}_g + 2\lambda \nabla^{\mathcal{S}(M)}_{\mathrm{grad}^g(\sigma)} s + w\Delta_g(\sigma)s + \lambda^2 |d\sigma|^2_g s), \\ \Delta^{\mathcal{S}(M)}_{\hat{g}} s =& e^{-2\sigma}(\Delta^{\mathcal{S}(M)}_g + (n-2)\nabla^{\mathcal{S}(M)}_{\mathrm{grad}^g(\sigma)})s, \\ \hat{J} =& e^{-2\sigma}(J - \Delta_g(\sigma) - \frac{n-2}{2}|d\sigma|^2_g). \end{aligned}$$

We have to prove three equations, where the last one is fulfilled already:

$$\begin{aligned} -\Box^{\mathcal{S}(M)}_{\hat{g},w}(e^{w\sigma} s) =& e^{(w-2)\sigma}\big[-\Box^{\mathcal{S}(M)}_{g,w} s - w_1 \,\mathrm{grad}^g(\sigma)^{\flat}(\nabla^{\mathcal{S}(M)} s)^{\natural}) - \frac{1}{2} ww_1 |d\sigma|^2_g s\big] \\ w_1(\nabla^{\mathcal{S}(M)}(e^{w\sigma} s))^{\hat{\natural}} =& e^{(w-2)\sigma}\big[w_1(\nabla^{\mathcal{S}(M)} s)^{\natural} + ww_1 \,\mathrm{grad}^g(\sigma) \otimes s\big] \\ ww_1 e^w s =& ww_1 e^w s. \end{aligned}$$

Note that $\cdot^{\natural}$ also depends on the metric $g \in c$. The second equation follows from

$$w_1(\nabla^{\mathcal{S}(M)}(e^{w\sigma} s))^{\hat{\natural}} = w_1 \sum_{i=1}^n \varepsilon_i \hat{s}_i \otimes \nabla^{\mathcal{S}(M)}_{\hat{s}_i}(e^{w\sigma} s)$$

$$
\begin{aligned}
&=e^{(w-2)\sigma} w_1 \big[\sum_{i=1}^{n} \varepsilon_i s_i \otimes w s_i(\sigma) s + \sum_{i=1}^{n} \varepsilon_i s_i \otimes \nabla^{\mathcal{S}(M)}_{s_i} s \big] \\
&=e^{(w-2)} \big[w w_1 \operatorname{grad}^g(\sigma) \otimes s + w_1 (\nabla^{\mathcal{S}(M)} s)^\natural \big].
\end{aligned}
$$

The first equation is more complicated: The left hand side is

$$
\begin{aligned}
\Box^{\mathcal{S}(M)}_{\hat{g},w}(e^{w\sigma} s) =& (\Delta^{\mathcal{S}(M)}_{\hat{g}} + w\hat{J})(e^{w\sigma} s) \\
=& e^{-2\sigma} \Big(\Delta^{\mathcal{S}(M)}_{g}(e^{w\sigma} s) + (n-2) \nabla^{\mathcal{S}(M)}_{\operatorname{grad}^g(\sigma)}(e^{w\sigma} s) \Big) \\
&+ w e^{-2\sigma} (J - \Delta_g(\sigma) - \frac{n-2}{2} |d\sigma|^2_g) e^{w\sigma} s \\
=& e^{-2\sigma} \Big(e^{w\sigma} \Delta^{\mathcal{S}(M)}_{g} s + 2 w e^{w\sigma} \nabla^{\mathcal{S}(M)}_{\operatorname{grad}^g(\sigma)} s + w \Delta_g(\sigma) \cdot s + w^2 |d\sigma|^2_g \cdot s \\
&+ (n-2) w e^{w\sigma} |d\sigma|^2_g \cdot s + (n-2) e^{w\sigma} \nabla^{\mathcal{S}(M)}_{\operatorname{grad}^g(\sigma)} s + w e^{w\sigma} J \cdot s \\
&- w e^{w\sigma} \Delta_g(\sigma) \cdot s - \frac{n-2}{2} w e^{w\sigma} |d\sigma|^2_g \cdot s \Big) \\
=& e^{(w-2)\sigma} \Big(\Delta^{\mathcal{S}(M)}_{g} s + w J \cdot s + (2w + n - 2) \nabla^{\mathcal{S}(M)}_{\operatorname{grad}^g(\sigma)} s \\
&+ (w^2 + \frac{n-2}{2} w |d\sigma|^2_g) \cdot s \Big).
\end{aligned}
$$

The right hand side is

$$
\begin{aligned}
&e^{(w-1)\sigma} \Big(- e^{-\sigma} \Box^{\mathcal{S}(M)}_{g,w} s - w_1 e^{-\sigma} \operatorname{grad}^g(\sigma)^\flat (\nabla^{\mathcal{S}(M)} s)^\natural + \frac{1}{2} w w_1 e^{-\sigma} |d\sigma|^2_g s \Big) \\
&= e^{(w-2)\sigma} \big[- \Box^{\mathcal{S}(M)}_{g,w} s - w_1 \nabla^{\mathcal{S}(M)}_{\operatorname{grad}^g(\sigma)} s - \frac{1}{2} w w_1 |d\sigma|^2_g s \big].
\end{aligned}
$$

This completes the proof, since $(w^2 + w\frac{n-2}{2}) = \frac{1}{2} w w_1$. $\Box$

In spirit of the tractor C-operator of the spinor bundle, we define the tractor C-operator of the standard spin tractor bundle by

$$
\begin{aligned}
C^{\mathcal{S}(M)}(g,w) :& \Gamma\big(\mathcal{S}^{\mathcal{T}}(M)_g\big) \to \Gamma\big(\mathcal{S}(M)\big) \\
C^{\mathcal{S}(M)}(g,w)(\tau_g \otimes e) :=& (n + w_1 n + w w_1) s_1 + (n + 2w) \operatorname{div}^{\otimes}(\eta) \\
&- (\Delta^{\mathcal{S}(M)}_g + (w - 1 - w_1) J) s_2,
\end{aligned}
$$

where $\Gamma\left(\mathcal{S}^{\mathcal{T}}(M)_g\right) \ni \tau_g \otimes e = (s_1, \eta, s_2)$. It satisfies the following conformal covariance of bi-degree $(w, w-1)$:

Lemma 5.20 *Let $\hat{g} = e^{2\sigma} g$ be conformally related to g, and let $\tau_g \otimes e \in \Gamma\big(\mathcal{S}^{\mathcal{T}}(M)_g\big)$.*

Then one has

$$C^{\mathcal{S}(M)}(\hat{g},w)(e^{w\sigma}T^{\otimes}(g,\sigma)(\tau_g\otimes e)) = e^{(w-1)\sigma}C^{\mathcal{S}(M)}(g,w)(\tau_g\otimes e),$$

i.e., $C^{\mathcal{S}(M)}(g,w)$ *is a conformally covariant differential operator of bi-degree* $(w, w-1)$.

Proof. To prove this lemma, we first recall the following formulas relating objects defined with respect to g and to $\hat{g}$:

$$\begin{aligned}\hat{\nabla}_X Y =& \nabla_X Y + X(\sigma)Y + Y(\sigma)X - g(X,Y)\,\mathrm{grad}^g(\sigma),\\ \hat{\mathrm{div}}^{\otimes}(Y\otimes s) =& \mathrm{div}^{\otimes}(Y\otimes s) + nY(\sigma)s,\end{aligned}$$

Using these we can first calculate in terms of $\tau_g\otimes s = (s_1,\eta,s_2)$ the argument of $C^{\mathcal{S}(M)}(\hat{g},w)$ to be equal to

$$e^{w\sigma}T^{\otimes}(g,\sigma)(\tau_g\otimes e) = \begin{pmatrix} e^{(w-1)\sigma}(s_1 - d\sigma(\eta) - \frac{1}{2}|d\sigma|^2_g s_2)\\ e^{(w-1)\sigma}(\eta + \mathrm{grad}^g(\sigma)\otimes s_2)\\ e^{(w+1)}\sigma s_2\end{pmatrix},$$

so that using the definition of the tractor C-operator we obtain

$$\begin{aligned}&C^{\mathcal{S}(M)}(\hat{g},w)\big(e^{w\sigma}T^{\otimes}(g,\sigma)(\tau_g\otimes e)\big)\\ &\qquad=(n+w_1n+ww_1)e^{(w-1)\sigma}(s_1 - d\sigma(\eta) - \frac{1}{2}|d\sigma|^2_g s_2)\\ &\qquad\quad+(n+2w)\hat{\mathrm{div}}^{\otimes}\big(e^{(w-1)\sigma}(\eta+\mathrm{grad}^g(\sigma)\otimes s_2)\big)\\ &\qquad\quad-(\Delta^{\mathcal{S}(M)}_{\hat{g}} + (w-1-w_1)\hat{J})e^{(w+1)\sigma}s_2.\end{aligned}$$

Now we are going to use the conformal transfomation laws given above and to sort the terms:

$$\begin{aligned}&C^{\mathcal{S}(M)}(\hat{g},w)\big(e^{w\sigma}T^{\otimes}(g,\sigma)(\tau_g\otimes e)\big)\\ &\qquad=(n+w_1n+ww_1)e^{(w-1)\sigma}s_1 - (n+w_1n+ww_1)e^{(w-1)\sigma}d\sigma(\eta)\\ &\qquad\quad+(n+2w)e^{(w-1)\sigma}\Big[(w-1)d\sigma(\eta) + \mathrm{div}^{\otimes}\eta + nd\sigma(\eta)\Big]\\ &\qquad\quad+(n+2w)e^{(w-1)\sigma}\Big[(w-1)d\sigma(\mathrm{grad}^g(\sigma))s_2\\ &\qquad\quad+\mathrm{div}^{\otimes}(\mathrm{grad}^g(\sigma)\otimes s_2) + nd\sigma(\mathrm{grad}^g(\sigma))s_2\Big]\\ &\qquad\quad-\frac{1}{2}(n+w_1n+ww_1)e^{(w-1)\sigma}|d\sigma|^2_g s_2\\ &\qquad\quad-e^{-2\sigma}(\Delta^{\mathcal{S}(M)}_g + (n-2)\nabla^{\mathcal{S}(M)}_{\mathrm{grad}^g(\sigma)})(e^{w+1)\sigma}s_2)\end{aligned}$$

$$- (w-1-w_1)e^{-2\sigma}e^{(w+1)\sigma}(J - \Delta_g(\sigma) - \frac{n-2}{2}|d\sigma|_g^2)s_2.$$

It remains to show that terms not belonging to $C^{\mathcal{S}(M)}(g,w)(\tau_g \otimes e)$ cancel each other. For terms involving s_1 there is nothing to cancel. The coefficients belonging to $d\sigma(\eta)$ vanish, since $-(n + w_1 n + ww_1) + (n+2w)(w-1+n) = 0$. We are left with terms involving s_2. Recall that $\operatorname{div}^{\otimes}(\operatorname{grad}^g(\sigma) \otimes s_2) = \Delta_g(\sigma)s_2 + \nabla^{\mathcal{S}(M)}_{\operatorname{grad}^g(\sigma)} s_2$ and $\operatorname{grad}^g(\sigma)(\sigma) = d\sigma(\operatorname{grad}^g(\sigma)) = |d\sigma|_g^2$. Collecting all terms involving s_2 gives by the product rule for the Bochner-Laplacian, given in a previous proof, the following identities:

$$\begin{aligned}
&(n+2w)(w-1+n)d\sigma(\operatorname{grad}^g(\sigma))s_2 + (n+2w)\Delta_g(\sigma)s_2 + (n+2w)\nabla^{\mathcal{S}(M)}_{\operatorname{grad}^g(\sigma)} s_2 \\
&\qquad - \frac{1}{2}(n + w_1 n + ww_1)|d\sigma|_g^2 s_2 - e^{-(w+1)\sigma}\Delta_g^{\mathcal{S}(M)}(e^{(w+1)\sigma}s_2) \\
&\qquad - e^{-(w+1)\sigma}(n-2)\nabla^{\mathcal{S}(M)}_{\operatorname{grad}^g(\sigma)}(e^{(w+1)\sigma}s_2) \\
&\qquad - (w-1-w_1)(J - \Delta_g(\sigma) - \frac{n-2}{2}|d\sigma|_g^2)s_2 \\
&\quad = \big((n+2w)(w-1+n) - \frac{1}{2}(n + w_1 n + ww_1) + (w-1-w_1)\frac{n-2}{2}\big)d\sigma(\operatorname{grad}^g(\sigma)) \\
&\qquad + \big((n+2w) + (w-1-w_1)\big)\Delta_g(\sigma)s_2 + (n+2w)\nabla^{\mathcal{S}(M)}_{\operatorname{grad}^g(\sigma)} s_2 \\
&\qquad - \Delta_g^{\mathcal{S}(M)}s_2 - 2(w+1)\nabla^{\mathcal{S}(M)}_{\operatorname{grad}^g(\sigma)} s_2 - (w+1)\Delta_g(\sigma)s_2 - (w+1)^2|d\sigma|_g^2 s_2 \\
&\qquad - (n-2)(w+1)\operatorname{grad}^g(\sigma)(\sigma)s_2 - (n-2)\nabla^{\mathcal{S}(M)}_{\operatorname{grad}^g(\sigma)} s_2 - (w-1-w_1)Js_2 \\
&\quad = \Big[(n+2w)(w-1+n) - \frac{1}{2}(n + w_1 n + ww_1) + (w-1-w_1)\frac{n-2}{2} \\
&\qquad - (w+1)^2 - (n-2)(w+1)\Big]d\sigma(\operatorname{grad}^g(\sigma)) \\
&\qquad + \big((n+2w) + (w-1-w_1) - (w+1)\big)\Delta_g(\sigma)s_2 \\
&\qquad + \big((n+2w) - 2(w+1) - (n-2)\big)\nabla^{\mathcal{S}(M)}_{\operatorname{grad}^g(\sigma)} s_2 \\
&\qquad - \Delta_g^{\mathcal{S}(M)}s_2 - (w-1-w_1)Js_2.
\end{aligned}$$

Calculating the individual coefficients shows that almost all of them vanish. The term above reduces to

$$-\Delta^{\mathcal{S}(M)}s_2 - (w-1-w_1)Js_2,$$

which shows that

$$\begin{aligned}
&e^{-(w-1)\sigma}C^{\mathcal{S}(M)}(\hat{g},w)\big(e^{w\sigma}T^{\otimes}(g,\sigma)(\tau_g \otimes e)\big) \\
&\qquad\qquad = (n + w_1 n + ww_1)s_1 + (n+2w)\operatorname{div}^{\otimes}(\eta) \\
&\qquad\qquad\quad - \Delta_g^{\mathcal{S}(M)}s_2 - (w-1-w_1)Js_2 \\
&\qquad\qquad = C^{\mathcal{S}(M)}(g,w)(\tau_g \otimes e).
\end{aligned}$$

This completes the proof. □

Remark 5.21 Let us assume that M has empty boundary. The tractor C-operator of the standard spin tractor bundle is related to the formal adjoint of the tractor D-operator of the standard spin tractor bundle with respect to the L^2-scalar products induced by the bundle metrics $g^{\mathcal{S}(M)}$ of the standard spin tractor bundle and $g^{\mathcal{T}(M)}$ of the standard tractor bundle: Again, observe that

$$g^{\mathcal{T}(M)}\left(D^{\mathcal{S}(M)}(g,w)s, \begin{pmatrix} s_1 \\ \eta \\ s_2 \end{pmatrix}\right)_{L^2} = g^{\mathcal{S}(M)}\left(s, \left(D^{\mathcal{S}(M)}(g,w)\right)^* \begin{pmatrix} s_1 \\ \eta \\ s_2 \end{pmatrix}\right)_{L^2}$$

for compactly supported sections $s, s_1, s_2 \in \Gamma_c(\mathcal{S}(M))$, $\eta \in \Gamma_c(TM \otimes \mathcal{S}(M))$ and

$$\left(D^{\mathcal{S}(M)}(g,w)\right)^* \begin{pmatrix} s_1 \\ \eta \\ s_2 \end{pmatrix} := -(\Delta_g^{\mathcal{S}(M)} + wJ)s_2 + ww_1 s_1 - w_1 \operatorname{div}^{\otimes}(\eta).$$

Thus we have that

$$C^{\mathcal{S}(M)}(g,w) \begin{pmatrix} s_1 \\ \eta \\ s_2 \end{pmatrix} = \left(D^{\mathcal{S}(M)}(g,1-n-w)\right)^* \begin{pmatrix} s_1 \\ \eta \\ s_2 \end{pmatrix},$$

which yields the desired relation.

Corollary 5.22 *One has $C^{\mathcal{S}(M)}(g,w-1) \circ D^{\mathcal{S}(M)}(g,w) = 0$ for all $w \in \mathbb{R}$.*

Remark 5.23 The two operators $D^{\mathcal{S}(M)}(g,w)$ and $C^{\mathcal{S}(M)}(g,w)$ can also be transported to the tractor level. Namely, let us set

$$\begin{aligned} D_g^{\mathcal{S}(M)}(w) : \Gamma(\mathcal{S}(M)) &\to \Gamma(\mathcal{S}^{\mathcal{T}}(M)) \\ \psi &\mapsto D_g^{\mathcal{S}(M)}(w)s := (\Phi^g \otimes id)^{-1} \circ D^{\mathcal{S}(M)}(g,w)s, \end{aligned}$$

as well as

$$\begin{aligned} C_g^{\mathcal{S}(M)}(w) : \Gamma(\mathcal{S}^{\mathcal{T}}(M)) &\to \Gamma(\mathcal{S}(M)) \\ s &\mapsto C_g^{\mathcal{S}(M)}(w)(\tau \otimes s) := C^{\mathcal{S}(M)}(g,w) \circ (\Phi^g \otimes id)(\tau \otimes s). \end{aligned}$$

The results of Lemma 5.19 and Lemma 5.20 are equivalent to

$$\begin{aligned} D_{\hat{g}}^{\mathcal{S}(M)}(w)(e^{w\sigma}s) &= e^{(w-1)\sigma} D_g^{\mathcal{S}(M)}(w)s, \\ C_{\hat{g}}^{\mathcal{S}(M)}(w)(e^{w\sigma}\tau \otimes s) &= e^{(w-1)\sigma} \circ C_g^{\mathcal{S}(M)}(w)(\tau \otimes s), \end{aligned}$$

where $\hat{g} = e^{2\sigma}g$, $s \in \Gamma(\mathcal{S}(M))$, $\tau \otimes s \in \Gamma(\mathcal{S}^{\mathcal{T}}(M))$ and $w \in \mathbb{R}$.

Finally, let us note that the operators $D_g^{\mathcal{S}(M)}(w)$ and $C_g^{\mathcal{S}(M)}(w)$ admit generalizations that act on tensor products $\mathcal{S}^{\otimes^k \mathcal{T}}(M) := \mathcal{S}(M) \otimes (\otimes^k \mathcal{T}(M))$. Their definitions are obvious extensions of the definitions of $D_g^{\mathcal{S}(M)}(w)$ and $C_g^{\mathcal{S}(M)}(w)$ by taking appropriate covariant derivatives. Let us denote these operators by $D_g^{\mathcal{S}(M),k}(w)$ as well as $C_g^{\mathcal{S}(M),k}(w)$. For any $k \in \mathbb{N}$ they fulfill the same conformal covariance as $D_g^{\mathcal{S}(M)}(w)$ and $C_g^{\mathcal{S}(M)}(w)$, which is the $k = 0$ case.

5.4 Conformally covariant differential operators hidden in the tractor machinery

The tractor D- and C-operators, defined in the Section 5.3, are the tools of the construction of higher order conformally covariant differential operators $P_{2N}^{\mathcal{S}(M)}(g)$ on the standard spin tractor bundle. These operators produce conformal powers of the Dirac operator. To achieve explicit formulas, we have to compute the $g-$metric representation of $P_{2N}^{\mathcal{S}(M)}(g)$ and then compose it with the tractor C- and tractor D-operator of the spinor bundle. We will perform the computation for the cases $N = 1, 2$. For higher values of N, the computations are hardly manageable by hand, but our algorithm remains valid for all $N \in \mathbb{N}$.

The conformal covariance of the Dirac- and Box-operator can be deduced directly from evaluating $D_g^{S(M,g)}(\eta)$ and $D_g^{\mathcal{S}(M)}(w)$ for appropriate η and w:

Corollary 5.24 *The Dirac operator $\not{D} : \Gamma(S(M,g)) \to \Gamma(S(M,g))$ is conformally covariant of bi-degree $(\frac{1-n}{2}, -\frac{1+n}{2})$, that is,*

$$\hat{\not{D}}(e^{\frac{1-n}{2}\sigma}\hat{\psi}) = e^{-\frac{1+n}{2}}F_\sigma(\not{D}\psi),$$

where $\hat{g} = e^{2\sigma}g$ and $\psi \in \Gamma(S(M,g))$.

The Box-operator $\Box_{g,\frac{2-n}{2}}^{\mathcal{S}(M)} : \Gamma(\mathcal{S}(M)) \to \Gamma(\mathcal{S}(M))$ is conformally covariant of bi-degree $(\frac{2-n}{2}, -\frac{2+n}{2})$, that is,

$$e^{\frac{2+n}{2}\sigma}\Box_{\hat{g},\frac{2-n}{2}}^{\mathcal{S}(M)}(e^{\frac{2-n}{2}\sigma}s) = \Box_{g,\frac{2-n}{2}}^{\mathcal{S}(M)}s,$$

where $\hat{g} = e^{2\sigma}g$ and $s \in \Gamma(\mathcal{S}(M))$.

Proof. Set $\eta = \frac{1-n}{2}$. The conformal covariance property of $D^{S(M,g)}(g,\eta)$, given in Remark 5.18, implies

$$\hat{\not{D}}(e^{\frac{1-n}{2}\sigma}F_\sigma(\psi)) = e^{-\frac{n+1}{2}\sigma}F_\sigma(\not{D}\psi).$$

Set $w = \frac{2-n}{2}$. Since $w_1 = (n-2+2w) = 0$ we get the following indentity from Lemma 5.19:

$$\begin{pmatrix} -\Box^{\mathcal{S}(M)}_{\hat{g},w}(e^{w\sigma}s) \\ 0 \\ 0 \end{pmatrix} = e^{(w-1)\sigma} \begin{pmatrix} e^{-\sigma} & -e^{-\sigma}\operatorname{grad}(\sigma)^\flat & -\frac{1}{2}e^{-\sigma}|d\sigma|_g^2 \\ 0 & e^{-\sigma} & e^{-\sigma}\operatorname{grad}(\sigma)\otimes \\ 0 & 0 & e^{\sigma} \end{pmatrix} \begin{pmatrix} -\Box^{\mathcal{S}(M)}_{g,w} s \\ 0 \\ 0 \end{pmatrix}.$$

The vanishing entries imply that

$$-\Box^{\mathcal{S}(M)}_{\hat{g},w}(e^{\frac{2-n}{2}\sigma}s) = -e^{-\frac{2+n}{2}\sigma}\Box^{\mathcal{S}(M)}_{g,w}s,$$

which proves the corollary. □

Remark 5.25 The second part of Corollary 5.24 can be generalized: For $k \in \mathbb{N}$, the operator

$$\Box^{\mathcal{S}^{\otimes^k}\mathcal{T}(M)}_{g,\frac{2-n}{2}} := (\Delta_g^{\mathcal{S}^{\otimes^k}\mathcal{T}(M)} + \frac{2-n}{2}J) : \Gamma(\mathcal{S}^{\otimes^k}\mathcal{T}(M)) \to \Gamma(\mathcal{S}^{\otimes^k}\mathcal{T}(M))$$

is conformal covariant of bi-degree $(\frac{2-n}{2}, -\frac{n+2}{2})$.

Let us translate the Box-operator several times to obtain a series of differential operators acting on the standard spin tractor bundle, compare [GP03]:

$$P_2^{\mathcal{S}(M)}(g) := \Box^{\mathcal{S}(M)}_{g,\frac{2-n}{2}}$$

and

$$\begin{aligned} P_{2N}^{\mathcal{S}(M)}(g) :=& C_g^{\mathcal{S}(M)}(-\frac{2(N-1)+n}{2}) \circ \ldots \circ C_g^{\mathcal{S}(M),N-1}(-\frac{2+n}{2}) \circ \\ &\circ \Box^{\mathcal{S}^{\otimes^{N-1}}\mathcal{T}(M)}_{g,\frac{2-n}{2}} \circ D_g^{\mathcal{S}(M),N-1}(\frac{4-n}{2}) \circ \ldots \circ D_g^{\mathcal{S}(M)}(\frac{2N-n}{2}). \end{aligned} \tag{5.7}$$

Their main property are:

Proposition 5.26 *Let $N \in \mathbb{N}$, and let $\hat{g} = e^{2\sigma}g$ be conformally equivalent to g. Then the operator $P_{2N}^{\mathcal{S}(M)}(g)$ is conformally covariant of bi-degree $(\frac{2N-n}{2}, -\frac{2N+n}{2})$, i.e.,*

$$e^{\frac{2N+n}{2}\sigma}P_{2N}^{\mathcal{S}(M)}(\hat{g})(e^{\frac{2N-n}{2}\sigma}s) = P_{2N}^{\mathcal{S}(M)}(g)s$$

for all $s \in \Gamma(\mathcal{S}(M))$, and its leading part is given by $c(n,N)(\Delta_g^{\mathcal{S}(M)})^N$, where

$$c(n,N) := (-1)^{N-1}\prod_{k=1}^{N-1}[k(2+2k-n)].$$

In case of even dimensional manifolds one has $P_{2N}^{\mathcal{S}(M)}(g) = 0$ for $N \geq \frac{n}{2}$.

Proof. The statement for $N = 1$ has been proven in Corollary 5.24. For $N > 1$ the assertion follows since $P_{2N}^{\mathcal{S}(M)}(g)$ is a composition of conformally covariant operators (tractor C-operators, tractor D-operators and the Box-operator). The weights are chosen in such a way that the result is a conformally covariant operator of bi-degree $(\frac{2N-n}{2}, -\frac{2N+n}{2})$. From

$$\Box_{g,\frac{2-n}{2}}^{\mathcal{S}^{\otimes^{N-1}}\mathcal{T}(M)} \circ D_g^{\mathcal{S}(M),N-1}(\frac{4-n}{2})t = \begin{pmatrix} A_1 t \\ A_2 t \\ 0 \end{pmatrix}$$

for any $t \in \Gamma\left(\mathcal{S}^{\otimes^{N-2}}\mathcal{T}(M)\right)$ and suitable operators A_1 and A_2, see the proof of Proposition 5.33, and from

$$C_g^{\mathcal{S}(M),N-1}(-\frac{2+n}{2})(t_1, \eta, 0) = (4-n)t_1 - 2\operatorname{div}^{\otimes}(\eta)$$

for any $(t_1, \eta, 0) \in \Gamma\left(\mathcal{S}^{\otimes^{N-1}}\mathcal{T}(M)\right)$, we may deduce that

$$\begin{aligned} P_{2N}^{\mathcal{S}(M)}(g)s =& (-1)^{N-1} \prod_{k=1}^{N-1} \left[n - 2n(k+1) + (2k+n)(k+1)\right] (\Delta_g^{\mathcal{S}(M)})^N s + \mathrm{LOT} \\ =& (-1)^{N-1} \prod_{k=1}^{N-1} \left[k(2+2k-n)\right] (\Delta_g^{\mathcal{S}(M)})^N s + \mathrm{LOT} \end{aligned}$$

for any $s \in \Gamma(\mathcal{S}(M))$. Here, the factor of $(-1)^{N-1}$ arises from the tractor D-operators whereas the tractor C-operators produce the remaining factors. In case of even dimensional manifolds and for $N \geq \frac{n}{2}$ one has that $c(n, N) = 0$, hence the operator becomes zero. $\square$

A direct consequence of Lemma 5.14, Lemma 5.15 and Proposition 5.26 is

Theorem 5.27 *Let $N \in \mathbb{N}$. The operator*

$$D_{2N+1}(g) := C^{S(M,g)}(g, -\frac{2N+n}{2}) \circ P_{2N}^{\mathcal{S}(M)}(g) \circ D^{S(M,g)}(g, \frac{2N-n+1}{2})$$

is a conformally covariant differential operator of bi-degree $\left(\frac{2N+1-n}{2}, -\frac{2N+1+n}{2}\right)$ on the spinor bundle $S(M,g)$ with leading term a constant multiple of $D\!\!\!/^{2N+1}$. In case of even dimensional manifolds one has for $N \geq \frac{n}{2}$ that $D_{2N+1}(g) = 0$.

Proof. The conformal covariance of $D_{2N+1}(g)$ is a consequence of the conformal covariance of their building blocks together with their well-chosen parameters. That the leading term of $D_{2N+1}(g)$ is formed by a constant multiple of $\not{D}^{2N+1}$ can be seen as follows: From equation (5.8) we obtain that

$$(\Delta^{g,\mathcal{S}(M)})^N = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

for suitable operators A_i acting on the spinor bundle $S(M,g)$. Thus, A_1 and A_4 are of order $2N$, whereas A_2 and A_3 are of order $2N-1$. Considering the composition

$$\begin{aligned} C^{S(M,g)}(g,-\frac{2N+n}{2}) &\circ (\Delta^{g,\mathcal{S}(M)})^N \circ D^{S(M,g)}(g,\frac{2N-n+1}{2}) \\ &= \begin{pmatrix} \frac{1}{2}\not{D} & N \end{pmatrix} \circ (\Delta^{g,\mathcal{S}(M)})^N \circ \begin{pmatrix} N \\ \frac{1}{2}\not{D} \end{pmatrix} \\ &= \frac{1}{2}N\not{D}A_1 + \frac{1}{2}NA_4\not{D} + \frac{1}{4}N\not{D}A_2\not{D} + N^2A_3 \end{aligned}$$

implies that the leading term is a constant multiple of $\not{D}^{2N+1}$, due to equation (2.3). Hence, the operator $D_{2N+1}(g)$ has leading term a constant multiple of $\not{D}^{2N+1}$, and this constant is given by a product of $c(n,N)$ and a factor depending on N arising by the composition above. In case of even dimensional manifolds we have that $D_{2N+1}(g) = 0$ for $N \geq \frac{n}{2}$. $\square$

In order to derive explicit formulas for a conformal third- and fifth power of the Dirac operator, we start to express the operators $P_2^{\mathcal{S}(M)}(g)$ and $P_4^{\mathcal{S}(M)}(g)$ in their $g-$metric representation.

Proposition 5.28 *The Box-operator $P_2^{\mathcal{S}(M)}(g) = \Box^{\mathcal{S}(M)}_{g,\frac{2-n}{2}}$, computed with respect to the splitting Ψ^g of the standard spin tractor bundle, is given by*

$$\Box^{g,\mathcal{S}(M)}_{\frac{2-n}{2}} := \Psi^g \circ \Box^{\mathcal{S}}_{g,\frac{2-n}{2}} \circ (\Psi^g)^{-1} = \begin{pmatrix} -\not{D}^2 & 2\not{D} \\ (P,\nabla^{S(M,g)}) + \frac{1}{2}\operatorname{grad}^g(J) & -\not{D}^2 \end{pmatrix},$$

where $\not{D}$ is the Dirac operator of (M,g) and $(P,\nabla^{S(M,g)}\psi)$ was introduced in equation (2.4).

Proof. Let $s \in \Gamma(\mathcal{S}(M))$ and recall the formula for the covariant derivative $\nabla^{g,\mathcal{S}(M)}$ in its $g-$metric representation,

$$\nabla_X^{g,\mathcal{S}(M)}\Psi^g s = \begin{pmatrix} \nabla_X^{S(M,g)} & X\cdot \\ \frac{1}{2}P(X)^\natural\cdot & \nabla_X^{S(M,g)} \end{pmatrix}\Psi^g s,$$

for all $X \in \mathfrak{X}(M)$. Applying this to $\nabla^{g,\mathcal{S}(M)}_{\nabla_{s_i}s_i}\Psi^g s$ and $\nabla^{g,\mathcal{S}(M)}_{s_i}\nabla^{g,\mathcal{S}(M)}_{s_i}$ we have

$$\nabla^{g,\mathcal{S}(M)}_{s_i}\nabla^{g,\mathcal{S}(M)}_{s_i} = \begin{pmatrix} \nabla^{S(M,g)}_{s_i}\nabla^{S(M,g)}_{s_i} + \frac{1}{2}s_i \cdot P(s_i)^\natural \cdot & \nabla_{s_i}s_i + s_i \cdot \nabla^{S(M,g)}_{s_i} \\ \frac{1}{2}P(s_i)^\natural \cdot \nabla^{S(M,g)}_{s_i} + \frac{1}{2}\nabla^{S(M,g)}(P(s_i)^\natural \cdot) & \nabla^{S(M,g)}_{s_i}\nabla^{S(M,g)}_{s_i} + \frac{1}{2}P(s_i)^\natural \cdot s_i \cdot \end{pmatrix},$$

so that

$$\Delta^{g,\mathcal{S}(M)}\Psi^g s = \begin{pmatrix} \Delta^{S(M,g)}_g - \frac{1}{2}J & 2\not{D} \\ (P, \nabla^{S(M,g)}) + \frac{1}{2}\operatorname{grad}^g(J) & \Delta^{S(M,g)}_g - \frac{1}{2}J \end{pmatrix}\Psi^g s, \tag{5.8}$$

since $\sum_i \varepsilon_i s_i \cdot P(s_i) \cdot = -J \cdot$. We use the Weitzenböck formula for the Dirac operator $\Delta^{S(M,g)}_g = -\not{D}^2 + \frac{1}{4}R = -\not{D}^2 + \frac{n-1}{2}J$, given in equation (2.3), in order to see that $\Box^{g,\mathcal{S}}_{\frac{2-n}{2}}\Psi^g s$ is equal to

$$\begin{pmatrix} -\not{D}^2 + \frac{n-2}{2}J & 2\not{D} \\ (P, \nabla^{S(M,g)}) + \frac{1}{2}\operatorname{grad}^g(J) & -\not{D}^2 + \frac{n-2}{2}J \end{pmatrix}\Psi^g s + \begin{pmatrix} \frac{2-n}{2}J & 0 \\ 0 & \frac{2-n}{2}J \end{pmatrix}\Psi^g s,$$

which completes the proof. □

Now, we are able to state:

Theorem 5.29 *Let (M, g) be a spin manifold. The operator $D_3(g)$ is given by*

$$D_3(g)\psi = -\frac{1}{2}\left[\not{D}^3\psi - (P, \nabla^{S(M,g)}\psi) - (\nabla^{S(M,g)}, P \cdot \psi)\right],$$

for any $\psi \in \Gamma(S(M, g))$.

Proof. We can compute

$$\begin{aligned} D_3(g)\psi =& C^{S(M,g)}_g(-\frac{2+n}{2}) \circ \Box^{g,\mathcal{S}(M)}_{\frac{2-n}{2}} \circ D^{S(M,g)}_g(\frac{3-n}{2})\psi \\ =& C^{S(M,g)}_g(-\frac{2+n}{2}) \circ \Box^{g,\mathcal{S}(M)}_{\frac{2-n}{2}} \begin{pmatrix} \psi \\ \frac{1}{2}\not{D}\psi \end{pmatrix} \\ =& C^{S(M,g)}_g(-\frac{2+n}{2}) \begin{pmatrix} 0 \\ (P, \nabla\psi) + \frac{1}{2}\operatorname{grad}^g(J) \cdot \psi - \frac{1}{2}\not{D}^3\psi \end{pmatrix} \\ =& -\frac{1}{2}\left[\not{D}^3\psi - 2(P, \nabla\psi) - \operatorname{grad}^g(J) \cdot \psi\right], \end{aligned}$$

which shows the desired form. □

Note that up to a factor of $-\frac{1}{2}$, this is exactly the third conformal power of the Dirac operator obtained in Proposition 4.18.

Following this concept we can compute a conformal fifth power of the Dirac operator. Firstly, we calculate the $g-$metric representation of $P_4^{\mathcal{S}(M)}(g)$. This leads to a decomposition $P_4^{\mathcal{S}(M)}(g) = P_4(g) + R(g)$, where $P_4(g), R(g) : \Gamma\left(\mathcal{S}(M)_g\right) \to \Gamma\left(\mathcal{S}(M)_g\right)$ are conformally covariant differential operators of bi-degree $(\frac{4-n}{2}, -\frac{4+n}{2})$. Note that this kind of decomposition did not appear in $P_2^{\mathcal{S}(M)}(g)$. Secondly, we translate $P_4(g)$ and $R(g)$ with the tractor D- and tractor C-operator of the spinor bundle, which gives two conformally covariant differential operators acting on the spinor bundle, one of fifth order, see Theorem 5.39, and one of order less or equal one, see Remark 5.40.

To do this, recall that we trivialize the bundle $\mathcal{T}(M) \otimes \mathcal{S}(M)$ only in the first factor by using a metric in the conformal class. Thus, we are able to represent the covariant derivative $\nabla^{\otimes}$ in matrix form:

Lemma 5.30 *With respect to a metric $g \in c$, the covariant derivative $\nabla^{\otimes}$ is given by*

$$\begin{aligned}\nabla_X^{g,\otimes}(\Phi^g\tau \otimes s) : &= \nabla_X^{g,\mathcal{T}(M)}\Phi^g\tau \otimes s + \Phi^g\tau \otimes \nabla_X^{\mathcal{S}(M)}s \\ &= \begin{pmatrix} \nabla_X^{\mathcal{S}(M)} & -P(X,\cdot) & 0 \\ X\otimes & \nabla_X^{TM\otimes\mathcal{S}(M)} & P(X)^\natural\otimes \\ 0 & -g(X,\cdot) & \nabla_X^{\mathcal{S}(M)} \end{pmatrix} \Phi^g\tau \otimes s,\end{aligned}$$

for $\Phi^g\tau \otimes s \in \Gamma\left(\mathcal{S}^{\mathcal{T}}(M)_g\right)$. In particular, one has

$$T^{\otimes}(g,\sigma)\nabla_X^{g,\otimes}(\Phi^g\tau \otimes s) = \nabla_X^{\hat{g},\otimes}\left(T^{\otimes}(g,\sigma)(\Phi^g\tau \otimes s)\right). \tag{5.9}$$

Proof. Let $\tau \otimes s \in \Gamma\left(\mathcal{S}^{\mathcal{T}}(M)\right)$ and denote by $\Phi^g\tau = (\alpha, Y, \beta)_g$ the $g-$metric representation of the standard tractor component. Then we have

$$\begin{aligned}\nabla_X^{g,\otimes}(\Phi^g\tau \otimes s) =& \nabla_X^{g,\mathcal{T}(M)}\Phi^g\tau \otimes s + \Phi^g\tau \otimes \nabla_X^{\mathcal{S}(M)}s \\ =& \begin{pmatrix} \nabla_X\alpha - P(X,Y) \\ \alpha\cdot X + \nabla_X Y + \beta\cdot P(X)^\natural \\ -g(X,Y) + \nabla_X\beta \end{pmatrix} \otimes s + \Phi^g\tau \otimes \nabla_X^{\mathcal{S}(M)}s.\end{aligned}$$

Using the isomorphism $\Theta_g(\tau \otimes s) = (\alpha s, Y \otimes s, \beta s)$ we conclude

$$\begin{aligned}\nabla_X^{g,\otimes}\Phi^g\tau \otimes s &= \begin{pmatrix} \alpha\cdot\nabla_X^{\mathcal{S}(M)}s \\ Y\otimes\nabla_X^{\mathcal{S}(M)}s \\ \beta\cdot\nabla_X^{\mathcal{S}(M)}s \end{pmatrix} + \begin{pmatrix} \nabla_X\alpha\cdot s - P(X,Y)\cdot s \\ \left(\alpha\cdot X + \nabla_X Y + \beta\cdot P(X)^\natural\right)\otimes s \\ -g(X,Y)\cdot s + \nabla_X\beta\cdot s \end{pmatrix} \\ &= \begin{pmatrix} \nabla_X^{\mathcal{S}(M)}(\alpha\cdot s) - P(X,Y)\cdot s \\ \alpha\cdot X\otimes s + \nabla_X^{TM\otimes\mathcal{S}(M)}(Y\otimes s) + \beta\cdot P(X)^\natural\otimes s \\ -g(X,Y)\cdot s + \nabla_X^{\mathcal{S}(M)}(\beta\cdot s) \end{pmatrix},\end{aligned}$$

and the proof is complete. □

As a consequence we can derive the $g-$metric representation $\Delta^{g,\mathcal{S}^{\mathcal{T}}(M)}$ of the Bochner-Laplacian $\Delta_g^{\mathcal{S}^{\mathcal{T}}(M)} : \Gamma\left(\mathcal{S}^{\mathcal{T}}(M)\right) \to \Gamma\left(\mathcal{S}^{\mathcal{T}}(M)\right)$:

Proposition 5.31 *The Bochner-Laplacian $\Delta_g^{\mathcal{S}^{\mathcal{T}}(M)}$ has the $g-$metric representation*

$$\begin{pmatrix} \Delta_g^{\mathcal{S}(M)} - J & dJ + 2\delta^{\nabla^{\mathcal{S}(M)}}(P\#^{\otimes}) & -|P|_g^2 \\ 2(\nabla^{\mathcal{S}(M)}(\cdot))^{\natural} & \Delta_g^{TM\otimes\mathcal{S}(M)} - 2(P\#^{\otimes})^{\natural} & dJ^{\natural} \otimes +2\operatorname{tr}_g\left(P^{\natural}\otimes\nabla^{\mathcal{S}(M)}\right) \\ -n & -2\operatorname{div}^{\otimes} & \Delta_g^{\mathcal{S}(M)} - J \end{pmatrix},$$

where $P\#^{\otimes}(Y\otimes s)(X) := P(X,Y)s$ for $Y\otimes s \in \Gamma\left(TM\otimes\mathcal{S}(M)\right)$ and

$$\operatorname{tr}_g\left(P^{\natural}\otimes\nabla^{\mathcal{S}(M)}\right) := \sum_i \varepsilon_i P(s_i)^{\natural}\otimes\nabla_{s_i}^{\mathcal{S}(M)},$$

for a local section $\{s_i\} : U \to \mathcal{P}^g$.

Note that a $1-$ form $\eta \in \Omega^1(M)$ acts on an element $Y\otimes s \in TM\otimes\mathcal{S}$ by $\eta(Y\otimes s) := \eta(Y)s$.

Proof. Let $\{s_i\}_{i=1}^n : U \to \mathcal{P}^g$ be a local section, and let $\{\sigma^i\}_{i=1}^n$ be its dual. The Bochner-Laplacian is given locally by

$$\Delta^{g,\mathcal{S}^{\mathcal{T}}(M)} = \sum_{i=1}^n \varepsilon_i\left[\nabla_{s_i}^{g,\otimes}\nabla_{s_i}^{g,\otimes} - \nabla_{\nabla_{s_i}s_i}^{g,\otimes}\right] =: \begin{pmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & C_9 \end{pmatrix}.$$

We will seperately calculate the two summands, hence determine the C's, using the formula for $\nabla^{g,\otimes}$, Lemma 5.30. For the first summand, writing

$$\nabla_{s_i}^{g,\otimes}\nabla_{s_i}^{g,\otimes} =: \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{pmatrix},$$

we have the entries

$$\begin{aligned} A_1 =& \nabla_{s_i}^{\mathcal{S}(M)}\nabla_{s_i}^{\mathcal{S}(M)} - P(s_i)(s_i\otimes), \\ A_2 =& -\nabla_{s_i}^{\mathcal{S}(M)}\left(P(s_i)(\cdot)\right) - P(s_i)\nabla_{s_i}^{TM\otimes\mathcal{S}(M)}, \\ A_3 =& -P(s_i)\left(P(s_i)^{\natural}\otimes\right) = -|P|_g^2, \\ A_4 =& s_i\otimes\nabla_{s_i}^{\mathcal{S}(M)} + \nabla_{s_i}^{TM\otimes\mathcal{S}(M)}(s_i\otimes), \\ A_5 =& -s_i\otimes P(s_i) + \nabla_{s_i}^{TM\otimes\mathcal{S}(M)}\nabla_{s_i}^{TM\otimes\mathcal{S}(M)} - P(s_i)^{\natural}\otimes\sigma^i(\cdot), \\ A_6 =& \nabla_{s_i}^{TM\otimes\mathcal{S}(M)}(P(s_i)^{\natural}\otimes) + P(s_i)^{\natural}\otimes\nabla_{s_i}^{\mathcal{S}(M)}, \end{aligned}$$

$$
\begin{aligned}
A_7 &= -\, g(s_i, s_i),\\
A_8 &= -\, \sigma^i(\nabla^{TM\otimes\mathcal{S}(M)}_{s_i}) - \nabla^{\mathcal{S}(M)}_{s_i}(\sigma^i(\cdot)),\\
A_9 &= -\, \sigma^i(P(s_i)^\natural \otimes\,) + \nabla^{\mathcal{S}(M)}_{s_i}\nabla^{\mathcal{S}(M)}_{s_i}.
\end{aligned}
$$

The second summand is calculated to be equal to

$$
\nabla^{g,\otimes}_{\nabla_{s_i}s_i} = \begin{pmatrix} \nabla^{\mathcal{S}(M)}_{\nabla_{s_i}s_i} & -P(\nabla_{s_i}s_i) & 0 \\ \nabla_{s_i}s_i\otimes & \nabla^{TM\otimes\mathcal{S}(M)}_{\nabla_{s_i}s_i} & P(\nabla_{s_i}s_i)^\natural\otimes \\ 0 & -g(\nabla_{s_i}s_i,\cdot) & \nabla^{\mathcal{S}(M)}_{\nabla_{s_i}s_i} \end{pmatrix} =: \begin{pmatrix} B_1 & B_2 & B_3 \\ B_4 & B_5 & B_6 \\ B_7 & B_8 & B_9 \end{pmatrix}.
$$

The first and the last entry $C_1 = \sum_i \varepsilon(A_1 - B_1)$ and $C_9 = \sum_i \varepsilon(A_9 - B_9)$ are clear, since

$$
P(s_i)(s_i \otimes s) = P(s_i, s_i)s = \sigma^i(P(s_i)^\natural \otimes s)
$$

for any $s \in \Gamma(\mathcal{S}(M))$.

For the next one, let $(Y \otimes s) \in \Gamma(TM \otimes \mathcal{S}(M))$. Then we have

$$
\begin{aligned}
\sum_i \varepsilon_i(A_2 - B_2)(Y\otimes s) &= -\sum_i \varepsilon_i[\nabla^{\mathcal{S}(M)}_{s_i}(P(s_i)(Y\otimes s)) - P(s_i)\nabla^{TM\otimes\mathcal{S}(M)}_{s_i}(Y\otimes s)]\\
&\quad + \sum_i \varepsilon_i P(\nabla_{s_i}s_i)(Y\otimes s)\\
&= -\, dJ(Y\otimes s) - 2\sum_i \varepsilon_i P(s_i)\nabla^{TM\otimes\mathcal{S}(M)}_{s_i}(Y\otimes s).
\end{aligned}
$$

Furthermore,

$$
\delta^{\nabla^{\mathcal{S}(M)}}(P\#^{\otimes}(Y\otimes s)) = -dJ(Y\otimes s) - \sum_i \varepsilon_i P(s_i)\nabla^{TM\otimes\mathcal{S}(M)}_{s_i}(Y\otimes s)
$$

implies $C_2 = \sum_i \varepsilon_i(A_2 - B_2) = dJ + 2\delta^{\nabla^{\mathcal{S}(M)}}(P\#^{\otimes})$.

The next identity is $C_5 = \sum_i \varepsilon_i(A_5 - B_5) = \Delta^{TM\otimes\mathcal{S}(M)}_g - 2(P\#^{\otimes})^\natural$, which follows from

$$
\begin{aligned}
-\sum_i \varepsilon_i s_i \otimes P(s_i)(Y\otimes s) - \sum_i \varepsilon_i P(s_i)^\natural \otimes \sigma^i(Y\otimes s) &= -\, 2P(Y)^\natural \otimes s\\
&= -\, 2\left(P\#^{\otimes}(Y\otimes s)\right)^\natural,
\end{aligned}
$$

for $(Y \otimes s) \in \Gamma(TM \otimes \mathcal{S}(M))$.

We go on with calculating $C_6 = \sum_i \varepsilon_i(A_6 - B_6)$. For $s \in \Gamma(\mathcal{S}(M))$ it holds that

$$
\begin{aligned}
&\sum_i \varepsilon_i[\nabla^{TM\otimes\mathcal{S}(M)}_{s_i}(P(s_i)^\natural \otimes s) + P(s_i)^\natural \otimes \nabla^{\mathcal{S}(M)}_{s_i}s - P(\nabla_{s_i}s_i)^\natural \otimes s]\\
&\quad = \sum_i \varepsilon_i(\nabla^{T^*M}_{s_i}P(s_i) - P(\nabla_{s_i}s_i)) \otimes s_g + 2\sum_i \varepsilon_i P(s_i) \otimes \nabla^{g,\mathcal{S}(M)}_{s_i}s_g
\end{aligned}
$$

$$= dJ^\natural \otimes s + 2\,\mathrm{tr}_g(P^\natural \otimes \nabla^{\mathcal{S}(M)} s).$$

Here we made use of $\delta^{\nabla^{LC}}(P) = -dJ$.

The remaining entries C_3, C_4, C_7 and C_8 can be derived easily, which completes the proof. $\square$

Corollary 5.32 *With respect to $g \in c$, the Box-operator $\square^{g,\mathcal{S}^{\mathcal{T}}(M)}_{\frac{2-n}{2}}$ has the $g-$metric representation*

$$\begin{pmatrix} \Delta_g^{\mathcal{S}(M)} - \frac{n}{2}J & dJ + 2\delta^{\nabla^{\mathcal{S}(M)}}(P\#^{\otimes}) & -|P|^2 \\ 2(\nabla^{\mathcal{S}(M)}(\cdot))^\natural & \Delta_g^{TM\otimes\mathcal{S}(M)} - 2(P\#^{\otimes})^\natural + \frac{2-n}{2}J & dJ^\natural \otimes +2\,\mathrm{tr}_g(P^\natural \otimes \nabla^{\mathcal{S}(M)}) \\ -n & -2\,\mathrm{div}^{\otimes} & \Delta_g^{\mathcal{S}(M)} - \frac{n}{2}J \end{pmatrix}.$$

An application of Weitzenböck formulas (5.4) and some computations imply an explicit formula for the operator $P_4^{\mathcal{S}(M)}(g) : \Gamma(\mathcal{S}(M)) \to \Gamma(\mathcal{S}(M))$.

Proposition 5.33 *Let $s \in \Gamma(\mathcal{S}(M))$. Then one has*

$$P_4^{\mathcal{S}(M)}(g)s = (4-n)A(g)s - 2\,\mathrm{div}^{\otimes}(B(g)s),$$

where the operators

$$A(g) : \Gamma(\mathcal{S}(M)) \to \Gamma(\mathcal{S}(M)),$$
$$B(g) : \Gamma(\mathcal{S}(M)) \to \Gamma(TM \otimes \mathcal{S}(M))$$

are given by

$$\begin{aligned} A(g)s =& -(\Delta_g^{\mathcal{S}(M)})^2 s + (n-2)J\Delta_g^{\mathcal{S}(M)} s + 4\delta^{\nabla^{\mathcal{S}(M)}}(P\#^{\otimes}(\nabla^{\mathcal{S}(M)}s)^\natural) \\ &+ (n-2)dJ((\nabla^{\mathcal{S}(M)}s)^\natural) + Q_{4,n}s, \\ B(g)s =& -2(\delta^{\nabla^{\mathcal{S}(M)}}(\mathcal{R}^{\mathcal{S}(M)}s))^\natural - 2(\mathrm{tr}_g)_{2,3}(\mathcal{R}^{\mathcal{S}(M)}\nabla^{\mathcal{S}(M)}s)^\natural \end{aligned}$$

for $Q_{4,n}s = \frac{n-4}{2}(-\frac{n}{2}J^2 + \Delta_g(J) + 2|P|_g^2)s$.

Proof. We will perform two calculations. Firstly, we have to compute $\square^{g,\mathcal{S}^{\mathcal{T}}(M)}_{\frac{2-n}{2}} \circ D(g, \frac{4-n}{2})s$. Secondly, we have to compose it with $C^{\mathcal{S}(M)}(g, -\frac{2+n}{2})$ to find $P_4^{\mathcal{S}(M)}(g)$. The first step needs the Weitzenböck formula for $1-$forms with values in the standard spin tractor bundle, which was given in equation (5.4). Let us set

$$\square^{g,\mathcal{S}^{\mathcal{T}}(M)}_{\frac{2-n}{2}} \begin{pmatrix} -\square^{\mathcal{S}(M)}_{g,\frac{4-n}{2}} s \\ 2(\nabla^{\mathcal{S}(M)} s)^{\natural} \\ (4-n)s \end{pmatrix} =: \begin{pmatrix} A(g)s \\ B(g)s \\ F(g)s \end{pmatrix},$$

and compute $A(g)s$, $B(g)s$ and $F(g)s$ using Corollary 5.32. First one has

$$F(g)s = n\square^{\mathcal{S}(M)}_{g,\frac{4-n}{2}} s - 4\,\mathrm{div}^{\otimes}\left((\nabla^{\mathcal{S}(M)} s)^{\natural}\right) + (4-n)\Delta^{\mathcal{S}(M)}_g s - (4-n)\frac{n}{2} J s = 0,$$

since $\mathrm{div}^{\otimes}\left((\nabla^{\mathcal{S}(M)} s)^{\natural}\right) = \Delta^{\mathcal{S}(M)}_g s$. Now we are going to simplify $A(g)$ and $B(g)$:

$$\begin{aligned} B(g)s =& -2(\nabla^{\mathcal{S}(M)}(\square^{\mathcal{S}(M)}_{g,\frac{4-n}{2}} s))^{\natural} + (\Delta^{TM\otimes\mathcal{S}(M)}_g - 2P\#^{\otimes} + \frac{2-n}{2} J)(2(\nabla^{\mathcal{S}(M)} s)^{\natural}) \\ & + \big(dJ^{\natural} \otimes +2\sum_i \varepsilon_i P(s_i)^{\natural} \otimes \nabla^{\mathcal{S}(M)}_{s_i}\big)((4-n)s) \\ =& 2\big[\Delta^{TM\otimes\mathcal{S}(M)}_g (\nabla^{\mathcal{S}(M)} s)^{\natural} - \big(\nabla^{\mathcal{S}(M)}(\Delta^{\mathcal{S}(M)}_g s)\big)^{\natural}\big] \\ & -(4-n)(\nabla J)^{\natural} s - (4-n)J(\nabla^{\mathcal{S}(M)} s)^{\natural} - 4P\#^{\otimes}(\nabla^{\mathcal{S}(M)} s)^{\natural} \\ & + (2-n)J(\nabla^{\mathcal{S}(M)} s)^{\natural} + (4-n)dJ^{\natural} \otimes s + 2(4-n)\sum_i \varepsilon_i P(s_i)^{\natural} \otimes \nabla^{\mathcal{S}(M)}_{s_i} s \\ =& 2\big[\Delta^{TM\otimes\mathcal{S}(M)}_g (\nabla^{\mathcal{S}(M)} s)^{\natural} - \big(\nabla^{\mathcal{S}(M)}(\Delta^{\mathcal{S}(M)}_g s)\big)^{\natural}\big] - 2J(\nabla^{\mathcal{S}(M)} s)^{\natural} \\ & - 4P\#^{\otimes}(\nabla^{\mathcal{S}(M)} s)^{\natural} + 2(4-n)\sum_i \varepsilon_i P(s_i)^{\natural} \otimes \nabla^{\mathcal{S}(M)}_{s_i} s. \end{aligned}$$

The equations (5.4) and (5.5) give us

$$\begin{aligned} B(g)s =& -2(\delta^{\nabla^{\mathcal{S}(M)}}(\mathcal{R}^{\mathcal{S}(M)} s))^{\natural} - 2\big(\sum_i \varepsilon_i \mathcal{R}^{\mathcal{S}(M)}(\cdot, s_i)\nabla^{\mathcal{S}(M)}_{s_i} s\big)^{\natural} \\ & + 2\big(\sum_i \varepsilon_i Ric(\cdot, s_i)\nabla^{\mathcal{S}(M)}_{s_i} s\big)^{\natural} - 2J(\nabla^{\mathcal{S}(M)} s)^{\natural} - 4P\#^{\otimes}(\nabla^{\mathcal{S}(M)} s)^{\natural} \\ & + 2(4-n)\sum_i \varepsilon_i P(s_i)^{\natural} \otimes \nabla^{\mathcal{S}(M)}_{s_i} s. \end{aligned}$$

Due to $\big(Ric(\cdot, s_i)\nabla^{\mathcal{S}(M)}_{s_i} s\big)^{\natural} = Ric(s_i)^{\natural} \otimes \nabla^{\mathcal{S}(M)}_{s_i} s$ and $Ric = (n-2)P + Jg$, several summands cancel each other, and we are left with

$$B(g)s = -2(\delta^{\nabla^{\mathcal{S}(M)}}(\mathcal{R}^{\mathcal{S}(M)} s))^{\natural} - 2(\mathrm{tr}_g)_{2,3}\left(\mathcal{R}^{\mathcal{S}(M)}\nabla^{\mathcal{S}(M)} s\right)^{\natural}.$$

Now we restrict our attention to the determination of $A(g)$. From matrix multiplication we get

$$A(g)s = -(\Delta^{\mathcal{S}(M)}_g - \frac{n}{2})J)(\Delta^{\mathcal{S}(M)}_g + \frac{4-n}{2} J)s + 2dJ\big((\nabla^{\mathcal{S}(M)} s)^{\natural}\big)$$

$$+ 4\delta^{\nabla^{\mathcal{S}(M)}}(P\#^{\otimes}(\nabla^{\mathcal{S}(M)}s)^{\natural}) - (4-n)|P|_g^2 s.$$

Since $\Delta_g^{\mathcal{S}(M)}(J\cdot s) = \Delta_g(J)s + J\Delta_g^{\mathcal{S}(M)}s + 2dJ\big((\nabla^{\mathcal{S}(M)}s)^{\natural}\big)$ we are left with

$$\begin{aligned}
A(g)s =& -(\Delta_g^{\mathcal{S}(M)})^2 s + \frac{n}{2}J\Delta_g^{\mathcal{S}(M)}s - \frac{4-n}{2}\Delta_g(J)s - \frac{4-n}{2}J\Delta_g^{\mathcal{S}(M)}s \\
&- (4-n)dJ\big((\nabla^{\mathcal{S}(M)}s)^{\natural}\big) + \frac{n(4+n)}{4}J^2 s + 2dJ\big((\nabla^{\mathcal{S}(M)}s)^{\natural}\big) \\
&+ 4\delta^{\nabla^{\mathcal{S}(M)}}(P\#^{\otimes}(\nabla^{\mathcal{S}(M)}s)^{\natural}) - (4-n)|P|_g^2 s \\
=& -(\Delta_g^{\mathcal{S}(M)})^2 s + (n-2)J\Delta_g^{\mathcal{S}(M)}s + 4\delta^{\nabla^{\mathcal{S}(M)}}(P\#^{\otimes}(\nabla^{\mathcal{S}(M)}s)^{\natural}) \\
&+ (n-2)dJ\big((\nabla^{\mathcal{S}(M)}s)^{\natural}\big) + Q_{4,n}s,
\end{aligned}$$

where we have defined $Q_{4,n}s := \frac{n-4}{2}(-\frac{n}{2}J^2 + \Delta_g(J) + 2|P|_g^2)s$. We have completed the main part of the calculations. Indeed, composing our vertical stack $(A(g)s, B(g)s, 0)$ with the tractor C-operator of the standard spin tractor bundle yields

$$C^{\mathcal{S}(M)}(g, -\frac{2+n}{2})\begin{pmatrix} A(g)s \\ B(g)s \\ 0 \end{pmatrix} = (4-n)A(g)s - 2\operatorname{div}^{\otimes}(B(g)s),$$

and the proof is complete. $\square$

Before we state our main result let us pay some attention to the operator

$$\begin{aligned}
B(g)s =& -2(\delta^{\nabla^{\mathcal{S}(M)}}(\mathcal{R}^{\mathcal{S}(M)}s))^{\natural} - 2(\mathrm{tr}_g)_{2,3}\left(\mathcal{R}^{\mathcal{S}(M)}\nabla^{\mathcal{S}(M)}s\right)^{\natural} \\
=& -2((\delta^{\nabla^{\mathcal{S}(M)}}\mathcal{R}^{\mathcal{S}(M)})s)^{\natural} - 4(\mathrm{tr}_g)_{2,3}\left(\mathcal{R}^{\mathcal{S}(M)}\nabla^{\mathcal{S}(M)}s\right)^{\natural},
\end{aligned}$$

which showed up in the composition of Box-operator and tractor D-operator. From the divergence of the spin tractor curvature, the $g-$metric representation of the operator $B(g)s_g = -2\sum_i \varepsilon_i s_i \otimes B(g)_i s_g$ is given by

$$B(g)_i := \begin{pmatrix} (B_i)_1 & 0 \\ (B_i)_3 & (B_i)_4 \end{pmatrix}, \tag{5.10}$$

for

$$\begin{aligned}
(B_i)_1 =& \frac{n-4}{2}C(s_i)\cdot + \sum_j \varepsilon_j W(s_i,s_j)\cdot\nabla_{s_j}^{S(M,g)}, \\
(B_i)_3 =& -\frac{1}{2}B(s_i)^{\natural}\cdot + \sum_j \varepsilon_j[C(s_i,s_j)^{\natural}\cdot\nabla_{s_j}^{S(M,g)} + \frac{1}{2}W(s_i,s_j)\cdot P(s_j)^{\natural}\cdot], \\
(B_i)_4 =& \frac{n-2}{2}C(s_i)\cdot + \sum_j \varepsilon_j W(s_i,s_j)\cdot\nabla_{s_j}^{S(M,g)}.
\end{aligned}$$

Corollary 5.34 *The operator $B(g) : \Gamma(\mathcal{S}(M)) \to \Gamma(TM \otimes \mathcal{S}(M))$ is conformally covariant of bi-degree $(\frac{n-4}{2}, -\frac{n+4}{2})$, i.e., one has*

$$B(\hat{g})(e^{\frac{4-n}{2}\sigma}s) = e^{-\frac{n+4}{2}\sigma}B(g)(s),$$

which is equivalent to

$$B(\hat{g})T^{\mathcal{S}(M)}(g,\sigma)(e^{\frac{4-n}{2}\sigma}s_g) = e^{-\frac{n+4}{2}\sigma}\big(\operatorname{id}\otimes T^{\mathcal{S}(M)}(g,\sigma)\big)\big(B(g)(s_g)\big), \tag{5.11}$$

where $\hat{g} = e^{2\sigma}g$ and where $s \in \Gamma(\mathcal{S}(M))$ and $s_g \in \Gamma(\mathcal{S}(M)_g)$.

Proof. This is a consequence of the conformal transformation law

$$e^{\frac{n+2}{2}\sigma}\Box^{\hat{g},\mathcal{S}^{\mathcal{T}}(M)}_{\frac{2-n}{2}} \circ D(\hat{g}, \frac{4-n}{2})s = T^{\otimes}(g,\sigma) \circ \Box^{g,\mathcal{S}^{\mathcal{T}}(M)}_{\frac{2-n}{2}} \circ D(g, \frac{4-n}{2})(e^{\frac{n-4}{2}\sigma}s)$$

and the fact enlighted in the proof of Proposition 5.33, that the lower slot of the operator $\Box^{g,\mathcal{S}^{\mathcal{T}}(M)}_{\frac{2-n}{2}} \circ D(g, \frac{4-n}{2})$ vanishes. Note that no explicit formula was needed.

An alternative proof of equation (5.11) uses the conformal transformation laws of Weyl, Cotton and Bach tensors as well as that of the spinor covariant derivative $\nabla^{S(M,g)}$. For example, one of the four equations to be checked is

$$\begin{aligned} e^{-\sigma}e^{\frac{n+4}{2}\sigma}\frac{n-4}{2}\hat{C}(\hat{s}_i)\hat{\cdot}(e^{\frac{1}{2}\sigma}F_\sigma(\psi)) + e^{-\sigma}e^{\frac{n+4}{2}\sigma}\sum_j \varepsilon_j\hat{W}(\hat{s}_i,\hat{s}_j)\hat{\cdot}\nabla^{S(M,\hat{g})}_{\hat{s}_j}(e^{\frac{1}{2}\sigma}\hat{\psi}) \\ = e^{\frac{1}{2}\sigma}F_\sigma\Big(\frac{n-4}{2}C(s_i)\cdot(e^{\frac{n-4}{2}\sigma}\psi) + \sum_j \varepsilon_j W(s_i,s_j)\cdot\nabla^{S(M,g)}_{s_j}(e^{\frac{n-4}{2}\sigma}\psi)\Big), \end{aligned}$$

which holds, due to the conformal transformations laws for the Weyl and Cotton tensor. Just note, that the equation involving the Bach tensor is more complicated. But we don't have to compute anything, since we have received this as a direct consequence mentioned at the beginning of the proof. □

Remark 5.35 We have promised in Chapter 3, that we will get the conformal transfomation law for the Bach tensor B^g associated to a semi Riemannian manifold (M,g) almost for free. This fact is encoded in the conformal covariance property of the operator $B(g) : \Gamma(\mathcal{S}(M)) \to \Gamma(TM \otimes \mathcal{S}(M))$. Namely, one of the four equations arising from equation (5.11) is the conformal transformation law for the Bach tensor.

Since the operator $P_4^{\mathcal{S}(M)}(g)$ is now well understood in terms of spin tractor data, we want to find its $g-$metric representation. To derive the latter, we introduce some notations:

$$(P^2\cdot\psi)(X) := ((P\cdot P)(X))^{\natural}\cdot\psi = P\,(P(X))^{\natural}\cdot\psi, \tag{5.12}$$

$$\begin{aligned}
W\cdot\mathcal{R}\cdot\psi :=& \sum_{i,j}\varepsilon_i\varepsilon_j W(s_i,s_j)\cdot\mathcal{R}(s_i,s_j)\cdot\psi,\\
(C,P\cdot)\psi :=& \sum_i \varepsilon_i C(s_i)\cdot P(s_i)^{\natural}\cdot\psi, \qquad (5.13)\\
(P,C\cdot)\psi :=& \sum_i \varepsilon_i P(s_i)^{\natural}\cdot C(s_i)\cdot\psi, \qquad (5.14)\\
C\cdot W\cdot\psi :=& \sum_{i,j}\varepsilon_i\varepsilon_j C(s_i,s_j)^{\natural}\cdot W(s_i,s_j)\cdot\psi,
\end{aligned}$$

where $\{s_i\} : U \to \mathcal{P}^g$ is a local section, $\psi \in \Gamma(S(M,g))$ and $X \in \mathfrak{X}(M)$. On the right hand sides, Clifford multiplications of vector fields and 2−forms with a spinor $\psi \in \Gamma(S(M,g))$ occur.

Proposition 5.36 *The $g-$metric representation of the operator $P_4^{\mathcal{S}(M)}(g)$ is given by*

$$P_4^{\mathcal{S}(M)}(g) = \begin{pmatrix} (n-4)\mathcal{D}_3 D\!\!\!/ + W\cdot\mathcal{R}\cdot & -4(n-4)\mathcal{D}_3 \\ E & (n-4)D\!\!\!/\mathcal{D}_3 + W\cdot\mathcal{R}\cdot \end{pmatrix},$$

where

$$\begin{aligned}
E :=& (n-4)\big[\mathcal{D}_3 D\!\!\!/^2 + D\!\!\!/^2\mathcal{D}_3 - 2D\!\!\!/^2 + 2\big((P^2,\nabla^{S(M,g)}) + (\nabla^{S(M,g)},P^2)\big)\big]\\
&- (n-4)(C,P) + 2(B,\nabla^{S(M,g)}) + C\cdot W\cdot + W\cdot C\cdot\,.
\end{aligned}$$

Proof. Let us start to compute, in detail, the matrix representation of the operator $P_4^{\mathcal{S}(M)}(g)$. Our calculations are pointwise at $x \in M$, thus, let $\{s_i\} : U \to \mathcal{P}^g$ be a local section, which is synchronous at x, and let $s \in \Gamma(\mathcal{S}(M))$. From

$$\begin{aligned}
A(g)s =& -(\Delta_g^{\mathcal{S}(M)})^2 s + (n-2)J\Delta_g^{\mathcal{S}(M)}s + 4\delta^{\nabla^{\mathcal{S}(M)}}\big(P\#^{\otimes}(\nabla^{\mathcal{S}(M)}s)^{\natural}\big)\\
&+ (n-2)dJ\big((\nabla^{\mathcal{S}(M)}s)^{\natural}\big) + Q_{4,n}s
\end{aligned}$$

we have to compute

$$\Delta^{g,\mathcal{S}(M)}s_g = \begin{pmatrix} -D\!\!\!/^2 + \frac{n-2}{2}J & 2D\!\!\!/ \\ (P,\nabla^{S(M,g)}) + \frac{1}{2}\operatorname{grad}^g(J)\cdot & -D\!\!\!/^2 + \frac{n-2}{2}J \end{pmatrix} s_g,$$

and

$$\begin{aligned}
(\Delta^{g,\mathcal{S}(M)})^2 s_g =& \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} s_g,\\
A_1 :=& D\!\!\!/^4 - \frac{n-2}{2}(D\!\!\!/^2 J + JD\!\!\!/^2) + (\frac{n-2}{2})^2 J^2
\end{aligned}$$

$$\begin{aligned}
&+2\not{D}((P,\nabla^{S(M,g)}))+\not{D}(\operatorname{grad}^g(J)\cdot),\\
A_2 :=&-4\not{D}^3+2(n-2)J\not{D}+(n-2)\operatorname{grad}^g(J)\cdot,\\
A_3 :=&-\big[(P,\nabla^{S(M,g)}\not{D}^2)+\not{D}^2((P,\nabla^{S(M,g)}))\big]\\
&-\frac{1}{2}\big[\operatorname{grad}^g(J)\cdot\not{D}^2+\not{D}^2\big(\operatorname{grad}^g(J)\cdot\big)\big]\\
&+(n-2)J(P,\nabla^{S(M,g)})+\frac{n-2}{2}P\big(\operatorname{grad}^g(J)\big)^\natural\cdot+\frac{n-2}{2}J\operatorname{grad}^g(J)\cdot\\
=&\frac{1}{2}\mathcal{D}_3\not{D}^2+\frac{1}{2}\not{D}^2\mathcal{D}_3-\not{D}^5+(n-2)J(P,\nabla^{S(M,g)})\\
&+\frac{n-2}{2}P\big(\operatorname{grad}^g(J)\big)^\natural\cdot+\frac{n-2}{2}J\operatorname{grad}^g(J)\cdot,\\
A_4 :=&\not{D}^4-\frac{n-2}{2}(\not{D}^2J+J\not{D}^2)+(\frac{n-2}{2})^2J^2\\
&+2(P,\nabla^{S(M,g)}\not{D})+\operatorname{grad}^g(J)\cdot\not{D}.
\end{aligned}$$

Recall that $\delta^{\nabla^{\mathcal{S}(M)}}\big(P\#^{\otimes}(Y\otimes s)\big)=-dJ(Y\otimes s)-\sum_i\varepsilon_iP(s_i)\big(\nabla^{TM\otimes\mathcal{S}(M)}_{s_i}(Y\otimes s)\big)$, as seen in the proof of Lemma 5.31, and that we can write $dJ\big((\nabla^{\mathcal{S}(M)})^\natural\big)=(dJ,\nabla^{\mathcal{S}(M)})=\nabla^{\mathcal{S}(M)}_{\operatorname{grad}^g(J)}$. Thus,

$$\delta^{\nabla^{\mathcal{S}(M)}}\big(P\#^{\otimes}(\nabla^{g,\mathcal{S}(M)}s_g)^\natural\big)=\begin{pmatrix}B_1 & B_2\\ B_3 & B_4\end{pmatrix}s_g,$$

where

$$\begin{aligned}
B_1 :=&-\nabla^{\mathcal{S}(M,g)}_{\operatorname{grad}(J)}-\sum_i\varepsilon_iP(s_i)\big(\nabla^{TM\otimes\mathcal{S}(M,g)}_{s_i}(\nabla^{S(M,g)})^\natural\big)+\frac{1}{2}|P|^2_g\\
B_2 :=&-\operatorname{grad}^g(J)-2(P,\nabla^{S(M,g)})\\
B_3 :=&-\frac{1}{2}P(\operatorname{grad}^g(J))-\sum_{i,j}\varepsilon_i\varepsilon_jP_{ij}P(s_i)^\natural\cdot\nabla^{S(M,g)}_{s_j}\\
&-\sum_{i,j}\varepsilon_i\varepsilon_j\frac{1}{2}P_{ij}\nabla_{s_i}(P(s_j)^\natural)\cdot\\
=&-\frac{1}{2}\big[(P^2,\nabla^{S(M,g)})+(\nabla^{S(M,g)},P^2\cdot)\big]\\
B_4 :=&-\nabla^{S(M,g)}_{\operatorname{grad}^g(J)}-\sum_i\varepsilon_iP(s_i)\big(\nabla^{TM\otimes S(M,g)}_{s_i}(\nabla^{S(M,g)})^\natural\big)+\frac{1}{2}|P|^2_g.
\end{aligned}$$

Another summand is

$$\nabla^{g,\mathcal{S}(M)}_{\operatorname{grad}^g(J)}s_g=\begin{pmatrix}\nabla^{S(M,g)}_{\operatorname{grad}^g(J)} & \operatorname{grad}^g(J)\cdot\\ \frac{1}{2}P\big(\operatorname{grad}^g(J)\big)^\natural\cdot & \nabla^{S(M,g)}_{\operatorname{grad}^g(J)}\end{pmatrix}s_g.$$

Finally, note that the $Q_{4,n}-$summand has only a non trivial diagonal. Writing

$$A(g)s_g = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} s_g,$$

we have to determine the C_i for $i = 1, \ldots, 4$. The second entry is given by

$$\begin{aligned} C_2 =& - A_2 + 2(n-2)J\not{D} + 4B_2 + (n-2)\,\mathrm{grad}^g(J) \\ =& 4\not{D}^3 - 2(n-2)J\not{D} - (n-2)\,\mathrm{grad}^g(J)\cdot +2(n-2)J\not{D} \\ & - 4\,\mathrm{grad}^g(J) - 8(P, \nabla^{S(M,g)}) + (n-2)\,\mathrm{grad}^g(J)\cdot \\ =& 4[\not{D}^3 - 2(P, \nabla^{S(M,g)}) - \mathrm{grad}^g(J)\cdot] = 4\mathcal{D}_3. \end{aligned}$$

Let us compute the first entry C_1. We have

$$\begin{aligned} C_1 =& - A_1 + (n-2)J(-\not{D}^2 + \frac{n-2}{2}J) + 4B_1 + (n-2)\nabla^{S(M,g)}_{\mathrm{grad}^g(J)} \\ & + \frac{n-4}{4}(-\frac{n}{2}J^2 + \Delta_g(J) + 2|P|^2_g). \end{aligned}$$

Firstly, let us pick some summands involved in C_1, namely,

$$-2\not{D}((P, \nabla^{S(M,g)}\psi)) - 4\sum_i \varepsilon_i P(s_i)(\nabla^{TM\otimes S(M,g)}_{s_i}(\nabla^{S(M,g)}\psi)^\natural),$$

which come from A_1 and B_1. Now, the point is to move the Dirac operator in the first summand to the right side, which gives a part of the operator $\mathcal{D}_3 \circ \not{D} = \not{D}^4 - 2(P, \nabla^{S(M,g)}\not{D}) - \mathrm{grad}^g(J)\cdot\not{D}$. Let us recall the equalities

$$\begin{aligned} \sum_i \varepsilon_i s_i \cdot \mathcal{R}^{S(M,g)}(s_i, X)\psi =& \frac{1}{2}Ric(X)^\natural \cdot \psi, \\ [\not{D}, \nabla^{S(M,g)}_X \psi] =& \frac{1}{2}Ric(X)^\natural \cdot \psi, \\ \not{D}(X\cdot\psi) =& H(X)\psi - X\cdot\not{D}\psi, \end{aligned}$$

where $H(X)\psi = \sum_i \varepsilon_i s_i \cdot \nabla^{LC}_{s_i} X \cdot \psi - 2\nabla^{S(M,g)}_X \psi$. Now we calculate

$$\begin{aligned} -2\sum_i & \varepsilon_i \not{D}(P(s_i)^\natural \cdot \nabla^{S(M,g)}_{s_i}\psi) - 4\sum_i \varepsilon_i P(s_i)(\nabla^{TM\otimes S(M,g)}_{s_i}(\nabla^{S(M,g)}\psi)^\natural) \\ =& - 2\sum_i \varepsilon_i H(P(s_i)^\natural)\nabla^{S(M,g)}_{s_i}\psi + 2\sum_i \varepsilon_i P(s_i)^\natural \cdot \not{D}(\nabla^{S(M,g)}_{s_i}\psi) \\ & - 4\sum_i \varepsilon_i P(s_i)(\nabla^{TM\otimes S(M,g)}_{s_i}(\nabla^{S(M,g)}\psi)^\natural) \\ =& - 2\sum_i \varepsilon_i H(P(s_i)^\natural)\nabla^{S(M,g)}_{s_i}\psi + 2\sum_i \varepsilon_i P(s_i)^\natural \cdot (\nabla^{S(M,g)}_{s_i}\not{D}\psi + \frac{1}{2}Ric(s_i)^\natural \cdot \psi) \end{aligned}$$

$$
\begin{aligned}
&- 4\sum_i \varepsilon_i P(s_i)(\nabla^{TM\otimes S(M,g)}_{s_i}(\nabla^{S(M,g)}\psi)^\natural)\\
=&- 2\sum_{i,j} \varepsilon_i\varepsilon_j s_j \cdot \nabla_{s_j} P(s_i)^\natural \cdot \nabla^{S(M,g)}_{s_i}\psi + 4\sum_i \varepsilon_i \nabla^{S(M,g)}_{P(s_i)^\natural}\nabla^{S(M,g)}_{s_i}\psi\\
&+ 2\sum_i \varepsilon_i P(s_i)^\natural \cdot (\nabla^{S(M,g)}_{s_i} \not{D}\psi + \frac{1}{2}Ric(s_i)^\natural \cdot \psi)\\
&- 4\sum_i \varepsilon_i P(s_i)(\nabla^{TM\otimes S(M,g)}_{s_i}(\nabla^{S(M,g)}\psi)^\natural)\\
=&- 2\sum_{i,j} \varepsilon_i\varepsilon_j s_j \cdot \nabla_{s_j} P(s_i)^\natural \cdot \nabla^{S(M,g)}_{s_i}\psi + 2(P, \nabla^{S(M,g)}\not{D}\psi)\\
&- (n-2)|P|^2_g\psi - J^2\psi.
\end{aligned}
$$

These equations show that the computation of C_4 is quite similar to that of C_1. Let us continue. The first summand yields a Cotton tensor derivate

$$
\begin{aligned}
-2\sum_{i,j}&\varepsilon_i\varepsilon_j s_j \cdot \nabla_{s_j} P(s_i)^\natural \cdot \nabla^{S(M,g)}_{s_i}\psi\\
=&- 2\sum_{i,j,l} \varepsilon_i\varepsilon_j\varepsilon_l \nabla_{s_j} P(s_i, s_l) s_j \cdot s_l \cdot \nabla^{S(M,g)}_{s_i}\psi\\
=&- 2\sum_{i,j<l} \varepsilon_i\varepsilon_j\varepsilon_l [\nabla_{s_j} P(s_i, s_l) - \nabla_{s_l} P(s_i, s_j)] s_j \cdot s_l \cdot \nabla^{S(M,g)}_{s_i}\psi + 2\nabla^{S(M,g)}_{\mathrm{grad}^g(J)}\psi\\
=&- 2\sum_{i,j<l} \varepsilon_i\varepsilon_j\varepsilon_l C(s_j, s_l, s_i) s_j \cdot s_l \cdot \nabla^{S(M,g)}_{s_i}\psi + 2\nabla^{S(M,g)}_{\mathrm{grad}^g(J)}\psi\\
=&- 2\sum_i \varepsilon_i C(s_i) \cdot \nabla^{S(M,g)}_{s_i}\psi + 2\nabla^{S(M,g)}_{\mathrm{grad}^g(J)}\psi,
\end{aligned}
$$

where we have considered the Cotton tensor C to be a $1-$form with values in the $2-$forms, which we can multiply with a spinor, see equation (2.2). A final ingredient to compute C_1 is given by

$$
\not{D}^2(J\psi) = H(\,\mathrm{grad}^g(J))\psi + J\not{D}^2\psi.
$$

Putting things together, we end up with

$$
\begin{aligned}
C_1 =&- \not{D}^4 + \frac{n-2}{2}(\not{D}^2(J\cdot) + J\not{D}^2) - (\frac{n-2}{2})^2 J^2 - 2\not{D}((P, \nabla^{S(M,g)}))\\
&- \not{D}(\mathrm{grad}^g(J)\cdot) - (n-2)J\not{D}^2 + \frac{(n-2)^2}{2}J^2 - 4\nabla^{S(M,g)}_{\mathrm{grad}^g(J)}\\
&- 4\sum_i \varepsilon_i P(s_i)(\nabla^{TM\otimes S(M,g)}_{s_i}(\nabla^{S(M,g)})^\natural) + 2|P|^2_g + (n-2)\nabla^{S(M,g)}_{\mathrm{grad}^g(J)}\\
&+ \frac{n-4}{2}(-\frac{n}{2}J^2 + \Delta_g(J) + 2|P|^2_g)\\
=&- \not{D}^4 + \frac{n-4}{2}H(\,\mathrm{grad}^g(J)) + 2(P, \nabla^{S(M,g)}\not{D}) + \mathrm{grad}^g(J) \cdot \not{D}
\end{aligned}
$$

$$
\begin{aligned}
&+ (n-4)\nabla^{S(M,g)}_{\operatorname{grad}^g(J)} - 2\sum_i \varepsilon_i C(s_i)\cdot \nabla^{S(M,g)}_{s_i}\psi + \frac{n-4}{2}\Delta_g(J) \\
=& - \mathcal{D}_3 D\!\!\!/ - 2\sum_i \varepsilon_i C(s_i)\cdot \nabla^{S(M,g)}_{s_i}\psi,
\end{aligned}
$$

due to $H\big(\operatorname{grad}^g(J)\big)\psi = \Delta_g(J)\psi - 2\nabla^{S(M,g)}_{\operatorname{grad}^g(J)}\psi$. The computation of the entry C_4 differs from that of C_1 only in A_4. But as seen above, this will have the effect that we find $D\!\!\!/\mathcal{D}_3$ inside C_4 instead of $\mathcal{D}_3 D\!\!\!/$. More precisely,

$$
\begin{aligned}
C_4 =& - A_4 + (n-2)J(-D\!\!\!/^2 + \frac{n-2}{2}J) + 4B_4 + (n-2)\nabla^{S(M,g)}_{\operatorname{grad}^g(J)} \\
&+ \frac{n-4}{2}(-\frac{n}{2}J^2 + \Delta_g(J) + 2|P|^2_g) \\
=& - D\!\!\!/\mathcal{D}_3 - 2\sum_i \varepsilon_i C(s_i)\cdot \nabla^{S(M,g)}_{s_i}.
\end{aligned}
$$

Thus, we are left with the entry C_3, which is given by

$$
\begin{aligned}
C_3 =& - A_3 + (n-2)J\big((P,\nabla^{S(M,g)}) + \frac{1}{2}\operatorname{grad}^g(J)\cdot\big) + 4B_3 + (n-2)\frac{1}{2}P\big(\operatorname{grad}^g(J)\big)^\natural\cdot \\
=& - \Big[\frac{1}{2}\mathcal{D}_3 D\!\!\!/^2 + \frac{1}{2}D\!\!\!/^2\mathcal{D}_3 - D\!\!\!/^5 + (n-2)\big(J(P,\nabla^{S(M,g)}) \\
&+ \frac{1}{2}P\big(\operatorname{grad}^g(J)\big)^\natural\cdot + \frac{1}{2}J\operatorname{grad}^g(J)\cdot\Big] + (n-2)J(P,\nabla^{S(M,g)}) + \frac{n-2}{2}J\operatorname{grad}^g(J)\cdot \\
&- 2\big((P^2,\nabla^{S(M,g)}) + (\nabla^{S(M,g)},P^2\cdot)\big) + \frac{n-2}{2}P\big(\operatorname{grad}^g(J)\big)^\natural \\
=& - \frac{1}{2}\mathcal{D}_3 D\!\!\!/^2 - \frac{1}{2}D\!\!\!/^2\mathcal{D}_3 + D\!\!\!/^5 - 2\big((P^2,\nabla^{S(M,g)}) + (\nabla^{S(M,g)},P^2\cdot)\big).
\end{aligned}
$$

Hence, we have computed the $g-$metric form of $A(g)$ in terms of spinor bundle data.
Recalling the $g-$metric form of $B(g)$, given in equations (5.10), leads to

$$
\begin{aligned}
\delta^{TM\otimes\mathcal{S}(M)}(B(g)s) =& -\sum_i \varepsilon g(s_i)(\nabla^{TM\otimes\mathcal{S}(M)}_{s_i}(B(g)s)) = 2\sum_i \varepsilon_i \nabla^{S(M,g)}_{s_i}(B(g)_i s) \\
=& \begin{pmatrix} D_1 & D_2 \\ D_3 & d_4 \end{pmatrix},
\end{aligned}
$$

where the entries D_i are obtained by matrix multiplication of the covariant derivative and the $g-$metric representation of $B(g)$ given above. We get

$$
\begin{aligned}
D_1\psi =& 2\sum_{-} \varepsilon_i\Big[\frac{n-4}{2}\nabla^{S(M,g)}_{s_i}(C(s_i)\cdot\psi) \\
&+ \sum_j \big(\varepsilon_j \nabla^{S(M,g)}_{s_i}(W(s_i,s_j)\cdot\nabla^{S(M,g)}_{s_j}\psi) + s_i\cdot C(s_i,s_j)^\natural\cdot\nabla^{S(M,g)}_{s_j}\psi)\Big] \\
=& - (n-4)\sum_i \varepsilon_i C(s_i)\cdot\nabla^{S(M,g)}_{s_i}\psi + \sum_{i\neq j}\varepsilon_i\varepsilon_j W(s_i,s_j)\cdot\mathcal{R}^{S(M,g)}(s_i,s_j)\psi,
\end{aligned}
$$

where we have used Lemma 2.6 and

$$2W(s_i,s_j)\cdot\nabla^{S(M,g)}_{s_i}\nabla^{S(M,g)}_{s_j}=W(s_i,s_j)\cdot\mathcal{R}^{S(M,g)}(s_i,s_j).$$

Analogously to the computations done before, the entry D_2 vanishes because of vanishing traces. D_4 yields a similar result as D_1:

$$\begin{aligned}D_4\psi=&2\sum_i\varepsilon_i\big[\frac{n-2}{2}\nabla^{S(M,g)}_{s_i}(C(s_i)\cdot\psi)+\sum_j\varepsilon_j\nabla^{S(M,g)}_{s_i}(W(s_i,s_j)\cdot\nabla^{S(M,g)}_{s_i}\psi)\big]\\=&-(n-4)\sum_i\varepsilon_iC(s_i)\cdot\nabla^{S(M,g)}_{s_i}\psi+\sum_{i\neq j}\varepsilon_i\varepsilon_jW(s_i,s_j)\cdot\mathcal{R}^{S(M,g)}(s_i,s_j)\psi.\end{aligned}$$

Finally, we are left with

$$\begin{aligned}D_3=&2\Big[\frac{1}{2}\sum_i\varepsilon_iP(s_i)^\natural\cdot(\frac{n-4}{2}C(s_i)\cdot+\sum_j\varepsilon_jW(s_i,s_j)\cdot\nabla^{S(M,g)}_{s_j})\\&+\sum_i\varepsilon_i\nabla^{S(M,g)}_{s_i}\Big(-\frac{1}{2}B(s_i)^\natural\cdot+\sum_j\varepsilon_jC(s_i,s_j)^\natural\cdot\nabla^{S(M,g)}_{s_j}\\&+\frac{1}{2}\sum_j\varepsilon_jW(s_i,s_j)\cdot P(s_j)^\natural\cdot\Big)\Big],\end{aligned}$$

which yields

$$\begin{aligned}D_3=&\frac{n-4}{2}\sum_i\varepsilon_iP(s_i)^\natural\cdot C(s_i)\cdot+\sum_{i,j}\varepsilon_i\varepsilon_jP(s_i)^\natural\cdot W(s_i,s_j)\cdot\nabla^{S(M,g)}_{s_j}+\delta(B)^\natural\cdot\\&-\sum_i\varepsilon_iB(s_i)^\natural\cdot\nabla^{S(M,g)}_{s_i}+2\sum_{i,j}\varepsilon_i\varepsilon_j(\nabla_{s_i}C)(s_i,s_j,\cdot)^\natural\cdot\nabla^{S(M,g)}_{s_j}\\&+2\sum_{i,j}\varepsilon_i\varepsilon_jC(s_i,s_j,\cdot)^\natural\cdot\nabla^{S(M,g)}_{s_i}\nabla^{S(M,g)}_{s_j}+\sum_{i,j}\varepsilon_i\varepsilon_j(\nabla_{s_i}W)(s_i,s_j)\cdot P(s_j)^\natural\cdot\\&+\sum_{i,j}\varepsilon_i\varepsilon_jW(s_i,s_j)\cdot(\nabla_{s_i}P)(s_j,\cdot)^\natural\cdot+\sum_{i,j}\varepsilon_i\varepsilon_jW(s_i,s_j)\cdot P(s_j)^\natural\cdot\nabla^{S(M,g)}_{s_i}.\end{aligned}$$

The summands

$$\begin{aligned}&\sum_{i,j}\varepsilon_i\varepsilon_j(P(s_i)^\natural\cdot W(s_i,s_j)\cdot\nabla^{S(M,g)}_{s_j}+W(s_i,s_j)\cdot P(s_j)^\natural\cdot\nabla^{S(M,g)}_{s_i})\\&\qquad\qquad=2\sum_{i,j,k,l}\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_lW_{ijlk}P_{ik}s_l\cdot\nabla^{S(M,g)}_{s_j}\end{aligned}$$

and $2\sum_{i,j}\varepsilon_i\varepsilon_j(\nabla_{s_i}C)(s_i,s_j,\cdot)^\natural\cdot\nabla^{S(M,g)}_{s_j}$ represent the Bach tensor combined with the covariant derivative, i.e., $2\sum_i\varepsilon_iB(s_i)^\natural\cdot\nabla^{S(M,g)}_{s_i}=2(B,\nabla^{S(M,g)})$. The divergence of the Bach tensor was given by $\delta(B)^\natural\cdot\psi=(n-4)\sum_{i,j}\varepsilon_i\varepsilon_jP_{ij}C(s_i,\cdot,s_j)^\natural\cdot\psi$. Other summands simplify to

$$\sum_{i,j}\varepsilon_i\varepsilon_j W(s_i,s_j)\cdot(\nabla_{s_i}P)(s_j,\cdot)^\natural\cdot=\frac{1}{2}\sum_{i,j}\varepsilon_i\varepsilon_j W(s_i,s_j)\cdot C(s_i,s_j,\cdot)^\natural\cdot,$$

$$2\sum_{i,j}\varepsilon_i\varepsilon_j C(s_i,s_j)^\natural\cdot\nabla^{S(M,g)}_{s_i}\nabla^{S(M,g)}_{s_j}\psi=\sum_{i,j}\varepsilon_i\varepsilon_j C(s_i,s_j)^\natural\cdot\mathcal{R}^{S(M,g)}(s_i,s_j)(\psi),$$

where the latter transforms, using the relation $W=\mathcal{R}+P\owedge g$, to

$$\begin{aligned}
&\sum_{i,j}\varepsilon_i\varepsilon_j C(s_i,s_j)^\natural\cdot\mathcal{R}^{S(M,g)}(s_i,s_j)(\psi)\\
&\quad=\frac{1}{2}\sum_{i,j}\varepsilon_i\varepsilon_j C(s_i,s_j)^\natural\cdot\mathcal{R}(s_i,s_j)\cdot\psi\\
&\quad=\frac{1}{2}\sum_{i,j}\varepsilon_i\varepsilon_j C(s_i,s_j)^\natural\cdot\big(W(s_i,s_j)\cdot\psi-(P\owedge g)(s_i,s_j)\cdot\psi\big)\\
&\quad=\frac{1}{2}\sum_{i,j}\varepsilon_i\varepsilon_j C(s_i,s_j)^\natural\cdot W(s_i,s_j)\cdot\psi+\sum_i\varepsilon_i C(s_i)\cdot P(s_i)^\natural\cdot\psi,
\end{aligned}$$

where we haved used the skew symmetry property of the Cotton tensor, the tracelessness of the Cotton and the Weyl tensor, and Clifford-multiplication rules. Finally, from the divergence of the Weyl tensor, see Proposition 2.3, we get

$$\sum_{i,j}\varepsilon_i\varepsilon_j(\nabla_{s_i}W)(s_i,s_j)\cdot P(s_j)^\natural\cdot\psi=-(n-3)\sum_i\varepsilon_i C(s_i)\cdot P(s_i)^\natural\cdot\psi.$$

From these observations we find

$$\begin{aligned}
D_3\psi=&-\frac{n-4}{2}\sum_i\varepsilon_i C(s_i)\cdot P(s_i)^\natural\cdot\psi+(B,\nabla^{S(M,g)})\psi\\
&+\frac{1}{2}\sum_{i,j}\varepsilon_i\varepsilon_j\big(C(s_i,s_j)^\natural\cdot W(s_i,s_j)+W(s_i,s_j)\cdot C(s_i,s_j)^\natural\big)\cdot\psi.
\end{aligned}$$

Hence, the $g-$metric respresentation of the operator $P_4^{S(M)}(g)$ is given by

$$\begin{aligned}
P_4^{S(M)}(g)=&(4-n)\begin{pmatrix}-\mathcal{D}_3\not{D}-2(C,\nabla^{S(M,g)}) & 4\mathcal{D}_3\\ E_1 & -\not{D}\mathcal{D}_3-2(C,\nabla^{S(M,g)})\end{pmatrix}\\
&+2\begin{pmatrix}-(n-4)(C,\nabla^{S(M,g)})+\frac{1}{2}W\cdot\mathcal{R}\cdot & 0\\ E_2 & -(n-4)(C,\nabla^{S(M,g)})+\frac{1}{2}W\cdot\mathcal{R}\cdot\end{pmatrix},
\end{aligned}$$

where we have set

$$E_1:=-\frac{1}{2}\mathcal{D}_3\not{D}^2-\frac{1}{2}\not{D}^2\mathcal{D}_3+\not{D}^5-2\big((P^2,\nabla^{S(M,g)})+(\nabla^{S(M,g)},P^2)\big)$$

$$E_2 = -\frac{n-4}{2}(C,P) + (B, \nabla^{S(M,g)}) + \frac{1}{2}C \cdot W + \frac{1}{2}W \cdot C.$$

This completes the proof. $\square$

From the $g-$metric representation of the operator $P_4^{\mathcal{S}(M)}(g)$, we can derive a splitting into two operators

$$P_4^{\mathcal{S}(M)}(g) = P_4(g) + R(g), \tag{5.15}$$

where the operator $R(g)$ is given by

$$R(g) = \begin{pmatrix} W \cdot \mathcal{R}\cdot & 0 \\ C \cdot W \cdot + W \cdot C \cdot & W \cdot \mathcal{R}\cdot \end{pmatrix}.$$

Note that, using Clifford multiplication rules and the tracelessness of the Weyl tensor, we get

$$\begin{aligned}\sum_{i,j} \varepsilon_i \varepsilon_j W(s_i, s_j) \cdot \mathcal{R}(s_i, s_j) \cdot &= \sum_{i,j} \varepsilon_i \varepsilon_j W(s_i, s_j) \cdot (W - (P \owedge g))(s_i, s_j) \\ &= \sum_{i,j} \varepsilon_i \varepsilon_j W(s_i, s_j) \cdot W(s_i, s_j) \cdot .\end{aligned}$$

Therefore, the operator $R(g)$ is given by

$$R(g) = \begin{pmatrix} W \cdot W \cdot & 0 \\ C \cdot W \cdot + W \cdot C \cdot & W \cdot W \cdot \end{pmatrix}.$$

We will call $P_4(g) : \Gamma(\mathcal{S}(M)_g) \to \Gamma(\mathcal{S}(M)_g)$ the main part of $P_4^{\mathcal{S}(M)}(g)$, since it is conformally covariant with the same bi-degree as $P_4^{\mathcal{S}(M)}(g)$:

Proposition 5.37 *Let g and $\hat{g} = e^{2\sigma} g$ be conformally equivalent. Then one has*

$$e^{\frac{n+4}{2}\sigma} R(\hat{g}) T^{\mathcal{S}(M)}(g,\sigma)(e^{\frac{4-n}{2}\sigma} s_g) = T^{\mathcal{S}(M)}(g,\sigma) R(g) s_g$$

for any $s_g \in \Gamma(\mathcal{S}(M)_g)$, i.e., the operator $R(g)$ is a conformally covariant differential operator of bi-degree $(\frac{4-n}{2}, -\frac{n+4}{2})$.

Proof. The statement of the proposition is equivalent to the following matrix equation:

$$\begin{aligned} &e^{\frac{n+4}{2}\sigma} \begin{pmatrix} \hat{W}\hat{\cdot}\hat{W}\hat{\cdot} & 0 \\ \hat{C}\hat{\cdot}\hat{W}\hat{\cdot} + \hat{W}\hat{\cdot}\hat{C}\hat{\cdot} & \hat{W}\hat{\cdot}\hat{W}\hat{\cdot} \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2}\sigma} F_\sigma & 0 \\ \frac{1}{2} e^{-\frac{1}{2}\sigma} F_\sigma(\mathrm{grad}^g(\sigma)\cdot) & e^{-\frac{1}{2}\sigma} F_\sigma \end{pmatrix} e^{\frac{4-n}{2}\sigma} s_g \\ &= \begin{pmatrix} e^{\frac{1}{2}\sigma} F_\sigma & 0 \\ \frac{1}{2} e^{-\frac{1}{2}\sigma} F_\sigma(\mathrm{grad}^g(\sigma)\cdot) & e^{-\frac{1}{2}\sigma} F_\sigma \end{pmatrix} \begin{pmatrix} W \cdot W \cdot & 0 \\ C \cdot W \cdot + W \cdot C \cdot & W \cdot W \cdot \end{pmatrix} s_g. \end{aligned}$$

Two of the equations we have to prove are consequences of the conformal transformation law of the Weyl tensor given in Remark 5.11, namely

$$\begin{aligned} e^{\frac{n+4}{2}\sigma}\hat{W}\hat{\cdot}\hat{W}\hat{\cdot}e^{\frac{4-n}{2}\sigma}e^{\pm\frac{1}{2}\sigma}\hat{\psi} =& e^{(\frac{8}{2}\pm\frac{1}{2})\sigma}\sum_{i,j}\varepsilon_i\varepsilon_j\hat{W}(\hat{s}_i,\hat{s}_j)\hat{\cdot}\hat{W}(\hat{s}_i,\hat{s}_j)\hat{\cdot}\hat{\psi} \\ =& e^{\pm\frac{1}{2}\sigma}F_\sigma(W\cdot W\cdot\psi). \end{aligned}$$

The third one is trivial. Finally, the fourth one holds, if the following equation is true:

$$\begin{aligned} & e^{\frac{n+4}{2}\sigma}(\hat{C}\hat{\cdot}\hat{W}\hat{\cdot}+\hat{W}\hat{\cdot}\hat{C}\hat{\cdot})e^{\frac{1}{2}\sigma}F_\sigma(e^{\frac{4-n}{2}\sigma}\psi)+e^{\frac{n+4}{2}\sigma}\frac{1}{2}e^{-\frac{1}{2}\sigma}\hat{W}\hat{\cdot}\hat{W}\hat{\cdot}F_\sigma(\mathrm{grad}^g(\sigma)\cdot e^{\frac{4-n}{2}\sigma}\psi) \\ &\qquad =\frac{1}{2}e^{-\frac{1}{2}\sigma}F_\sigma(\mathrm{grad}^g(\sigma)\cdot W\cdot W\cdot\psi)+e^{-\frac{1}{2}\sigma}F_\sigma((C\cdot W\cdot+W\cdot C\cdot)\psi). \end{aligned}$$

To see the equality, look at

$$\begin{aligned} & e^{\frac{n+4}{2}\sigma}(\hat{C}\hat{\cdot}\hat{W}\hat{\cdot}+\hat{W}\hat{\cdot}\hat{C}\hat{\cdot})e^{\frac{1}{2}\sigma}F_\sigma(e^{\frac{4-n}{2}\sigma}\psi) \\ &\quad =e^{4\sigma}\sum_{i,j}\varepsilon_i\varepsilon_j(\hat{C}(\hat{s}_i,\hat{s}_j)^{\hat{\natural}}\hat{\cdot}\hat{W}(\hat{s}_i,\hat{s}_j)\hat{\cdot}+\hat{W}(\hat{s}_i,\hat{s}_j)\hat{\cdot}\hat{C}(\hat{s}_i,\hat{s}_j)^{\hat{\natural}}\hat{\cdot})e^{\frac{1}{2}\sigma}F_\sigma(\psi) \\ &\quad =e^{-\frac{1}{2}\sigma}F_\sigma(C\cdot W\cdot\psi+W\cdot C\cdot\psi) \\ &\qquad -e^{-\frac{1}{2}\sigma}F_\sigma(W(\cdot,\cdot,\mathrm{grad}^g(\sigma),\cdot)^\natural\cdot W\cdot\psi+W\cdot W(\cdot,\cdot,\mathrm{grad}^g(\sigma),\cdot)^\natural\cdot\psi), \end{aligned}$$

and the relation

$$\begin{aligned} & \frac{1}{2}W\cdot W\cdot\mathrm{grad}^g(\sigma)\cdot-\frac{1}{2}\,\mathrm{grad}^g(\sigma)\cdot W\cdot W\cdot \\ &\qquad =W(\cdot,\cdot,\mathrm{grad}^g(\sigma),\cdot)^\natural\cdot W+W\cdot W(\cdot,\cdot,\mathrm{grad}^g(\sigma),\cdot)^\natural\cdot\,. \end{aligned}$$

This completes the proof. $\square$

Remark 5.38 The conformal covariance of the operator $R(g)$ follows from an invariance argument as well. Note that

$$R(g)=4\sum_{i,j}\varepsilon_i\varepsilon_j\mathcal{R}^{g,\mathcal{S}(M)}(s_i,s_j)\circ\mathcal{R}^{g,\mathcal{S}(M)}(s_i,s_j).$$

From the conformal invariance of the spin tractor curvature, i.e., for $\hat{g}=e^{2\sigma}g$ we have $\mathcal{R}^{\hat{g},\mathcal{S}(M)}(X,Y)T^{\mathcal{S}(M)}(g,\sigma)=T^{\mathcal{S}(M)}(g,\sigma)\mathcal{R}^{g,\mathcal{S}(M)}(X,Y)$ for all $X,Y\in\mathfrak{X}(M)$, it follows the conformal covariance of the operator $R(g)$.

Now the goal is to compose the main part $P_4(g)$ of the operator $P_4^{\mathcal{S}(M)}(g)$ with the tractor C- and D-operator of the spinor bundle in order to get the reduced

$$D_5^{red}(g):=C_g^{S(M,g)}(-\frac{4-n}{2})\circ P_4(g)\circ D_g^{S(M,g)}(\frac{5-n}{2}).$$

This will yield a conformal fifth power of the Dirac operator acting on the spinor bundle.

Theorem 5.39 *Let (M,g) be a spin manifold and let $n \neq 4$. The operator $D_5^{red}(g)$ is given by*

$$\begin{aligned} D_5^{red}(g) =& (n-4)\Big[\not{D}\mathcal{D}_3\not{D} + 2(\not{D}^2\mathcal{D}_3 + \mathcal{D}_3\not{D}^2) - 4\not{D}^5 \\ &+ 8(P^2, \nabla^{S(M,g)}) + 8(\nabla^{S(M,g)}, P^2\cdot) + \frac{8}{n-4}(B, \nabla^{S(M,g)}) - 4(C,P)\Big]. \end{aligned}$$

Proof. First recall the tractor C- and D-operator of the spinor bundle and the operator

$$P_4(g) = \begin{pmatrix} (n-4)\mathcal{D}_3\not{D} & -4(n-4)\mathcal{D}_3 \\ \begin{matrix}(n-4)\big[\frac{1}{2}\mathcal{D}_3\not{D}^2 + \frac{1}{2}\not{D}^2\mathcal{D}_3 - \not{D}^5 \\ +2\big((P^2,\nabla^{S(M,g)}) + (\nabla^{S(M,g)}, P^2\cdot)\big) \\ +\frac{2}{n-4}(B,\nabla^{S(M,g)}) - (C,P)\cdot\big]\end{matrix} & (n-4)\not{D}\mathcal{D}_3 \end{pmatrix}.$$

Now we compute for $\psi \in \Gamma\big(S(M,g)\big)$:

$$\begin{aligned} C_g^{S(M,g)}(-\frac{4+n}{2}) \circ P_4(g) \circ D_g^{S(M,g)}(\frac{5-n}{2})\psi & \\ &= \begin{pmatrix}\frac{1}{2}\not{D} & 2\,\mathrm{id}\end{pmatrix} P_4(g) \begin{pmatrix} 2\psi \\ \frac{1}{2}\not{D}\psi\end{pmatrix} \\ &= (n-4)\big[\not{D}\mathcal{D}_3\not{D}\psi + 2(\not{D}^2\mathcal{D}_3\psi + \mathcal{D}_3\not{D}^2\psi) - 4\not{D}^5\psi \\ &\quad + 8(P^2, \nabla^{S(M,g)}\psi) + 8(\nabla^{S(M,g)}, P^2\cdot\psi) \\ &\quad + \frac{8}{n-4}(B, \nabla^{S(M,g)}\psi) - 4(C,P)\cdot\psi\big]. \end{aligned}$$

This completes the proof. $\square$

Remark 5.40 For completeness, one has to consider the composition of the conformal covariant operator $R(g)$ with the tractor C- and D-operator of the spinor bundle:

$$\begin{aligned} C_g^{S(M,g)}(-\frac{4+n}{2}) \circ R(g) \circ D_g^{S(M,g)}(\frac{5-n}{2})\psi & \\ &= \begin{pmatrix}\frac{1}{2}\not{D} & 2\,\mathrm{id}\end{pmatrix} R(g) \begin{pmatrix} 2\psi \\ \frac{1}{2}\not{D}\psi\end{pmatrix} \\ &= \not{D}(W\cdot W\cdot) + 4(C\cdot W + W\cdot C) + W\cdot W\cdot\not{D}. \end{aligned}$$

This is a conformally covariant differential operator with leading term $W\cdot W\cdot D\!\!\!/$. We could add the above term to the operator $D_5^{red}(g)$ to end up with the conformally covariant differential operator $D_5(g)$ of bi-degree $(\frac{5-n}{2},\frac{5+n}{2})$.

In what will follow we will also denote the operator $\frac{1}{n-4}D_5^{red}(g)$ by $\mathcal{D}_5$, for $n\neq 4$, but we have to point out that there is no proof that $\frac{1}{n-4}D_5^{red}(g)$ is the conformal fifth power of the Dirac operator obtained from the spectral theoretical construction, see Theorem 4.17.

In order to prove the formal self-adjointness of our conformal third- and fifth power of the Dirac operator, let us rewrite

$$\begin{aligned}\frac{8}{n-4}(B,\nabla^{S(M,g)}\psi)-4(C,P)\cdot\psi=&\frac{4}{n-4}\big[(B,\nabla^{S(M,g)})+(\nabla^{S(M,g)},B\cdot)\big]\\&-2\big[(C,P\cdot)+(P,C\cdot)\big].\end{aligned}$$

Hence, the operator $\mathcal{D}_5:\Gamma\big(S(M,g)\big)\to\Gamma\big(S(M,g)\big)$ takes the form

$$\begin{aligned}\mathcal{D}_5\psi=&D\!\!\!/\mathcal{D}_3D\!\!\!/\psi+2(D\!\!\!/^2\mathcal{D}_3\psi+\mathcal{D}_3D\!\!\!/^2\psi)-4D\!\!\!/^5\psi+4(2P^2+\frac{1}{n-4}B,\nabla^{S(M,g)}\psi)\\&+4(\nabla^{S(M,g)},2P^2\cdot\psi+\frac{1}{n-4}B\cdot\psi)-2\big[(C,P\cdot\psi)+(P,C\cdot\psi)\big].\end{aligned}$$

We can state the following:

Theorem 5.41 *Let (M,g) be a spin manifold without boundary. The operators $\mathcal{D}_3$ and $\mathcal{D}_5$ on $\Gamma_c\big(S(M,g)\big)$ are formally self-adjoint (anti self-adjoint) with respect to the L^2-scalar product on the spinor bundle.*

Proof. Recall that the operators $\mathcal{D}_3$ and $\mathcal{D}_5$ can be written as

$$\begin{aligned}\mathcal{D}_3=&D\!\!\!/^3-(P,\nabla^{S(M,g)})-(\nabla^{S(M,g)},P\cdot),\\\mathcal{D}_5=&D\!\!\!/\mathcal{D}_3D\!\!\!/+2[D\!\!\!/^2\mathcal{D}_3+\mathcal{D}_3D\!\!\!/^2]-4D\!\!\!/^5+8\big[(P^2,\nabla^{S(M,g)})+(\nabla^{S(M,g)},P^2\cdot)\big]\\&+\frac{4}{n-4}\big[(B,\nabla^{S(M,g)})+(\nabla^{S(M,g)},B\cdot)\big]-2\big[(C,P\cdot)+(P,C\cdot)\big].\end{aligned}$$

The bracket notation and Proposition 2.5 ensures the formal self-adjointness of $\mathcal{D}_3$. The formal self-adjointness of $\mathcal{D}_5$ follows from the formal self-adjointness of $D\!\!\!/$ and $\mathcal{D}_3$, Proposition 2.5, and the fact that

$$\begin{aligned}<(C,P\cdot)\phi+(P,C\cdot)\phi,\psi>=&\sum_i\varepsilon_i<C(s_i)\cdot P(s_i)^\natural\cdot\phi+P(s_i)^\natural\cdot C(s_i)\cdot\phi,\psi>\\=&(-1)^p\sum_i\varepsilon_i<\phi,C(s_i)\cdot P(s_i)^\natural\cdot\psi+P(s_i)^\natural\cdot C(s_i)\cdot\psi>\\=&(-1)^p<\phi,(C,P\cdot)\psi+(P,C\cdot)\psi>.\end{aligned}$$

This completes the proof. □

Let us close this section with an example.

Example 5.42 Let (S^n, g_c) be the sphere equipped with the round metric. The Schouten tensor P is equal to $\frac{1}{2}g_c$. Hence, the conformal third- and fifth power of the Dirac operator on the sphere are given by

$$\begin{aligned}
\mathcal{D}_3 =& \not{D}^3 - (P, \nabla) - (\nabla, P\cdot) \\
=& \not{D}^3 - \not{D} \\
=& (\not{D} - 1) \circ \not{D} \circ (\not{D} + 1), \\
\mathcal{D}_5 =& \not{D}\mathcal{D}_3\not{D} + 2[\not{D}^2\mathcal{D}_3 + \mathcal{D}_3\not{D}^2] - 4\not{D}^5 + 8\left[(P^2, \nabla) + (\nabla, P^2\cdot)\right] \\
=& \not{D}^5 - 5\not{D}^3 + 4\not{D} \\
=& (\not{D} - 2) \circ (\not{D} - 1) \circ \not{D} \circ (\not{D} + 1) \circ (\not{D} + 2),
\end{aligned}$$

which coincide with results obtained in [ES10] for conformal odd powers of the Dirac operator on the sphere with round metric.

5.5 Some other conformally covariant differential operators acting on the standard spin tractor bundle

This section is devoted to a new construction of conformally covariant differential operators on the standard spin tractor bundle. Let $k \in \mathbb{N}$ be odd and let us assume the existence of a conformally covariant differential operator $D_k(g) : \Gamma\left(S(M,g)\right) \to \Gamma\left(S(M,g)\right)$ of bi-degree $(\frac{k-n}{n}, -\frac{k+n}{2})$. By composition with the tractor D-operator $D^{S(M,g)}(g,\eta)$ and tractor C-operator $C^{S(M,g)}(g,\eta')$ for appropriate η, η', we will derive new conformally covariant differential operators $L_k(g)$ acting on the standard spin tractor bundle. These operators were found by an ad hoc computation analyzing $P_2^{\mathcal{S}(M)}(g)$ and its conformal covariance. Realizing the structure of these operators leads to a new way of presenting them. Let us now present the $L_k(g)$ in the form we found them first. Then, at the end of this section, in Remark 5.47, we will see that all we have actually done, is part of an underlying structure.

Recall the map $T^{\mathcal{S}(M)}(g,\sigma)$ given in equation (5.2). Abstractly, we want to define a metric depending differential operator $L_1(g) : \Gamma(\mathcal{S}(M)_g) \to \Gamma(\mathcal{S}(M)_g)$ with the property

$$L_1(\hat{g}) \circ e^{\frac{2-n}{2}\sigma} T^S(g,\sigma) = e^{-\frac{2+n}{2}\sigma} T^S(g,\sigma) \circ L_1(g) \tag{5.16}$$

for conformally related metrics $\hat{g} = e^{2\sigma}g$. Therefore, we make for some unknown differential operators $NW(g), \ldots, SE(g)$ that act on the spinor bundle $S(M,g)$, an ansatz

$$L_1(g) := \begin{pmatrix} NW(g) & NE(g) \\ SW(g) & SE(g) \end{pmatrix}.$$

Equation (5.16) is equivalent to

$$\begin{aligned}\begin{pmatrix} NW(\hat{g}) & NE(\hat{g}) \\ SW(\hat{g}) & SE(\hat{g}) \end{pmatrix} e^{\frac{2-n}{2}\sigma} \begin{pmatrix} e^{\frac{1}{2}\sigma}F_\sigma & 0 \\ \frac{1}{2}e^{-\frac{1}{2}\sigma}F_\sigma(\mathrm{grad}^g(\sigma)\cdot) & e^{-\frac{1}{2}\sigma}F_\sigma \end{pmatrix} \\ = e^{-\frac{2+n}{2}\sigma} \begin{pmatrix} e^{\frac{1}{2}\sigma}F_\sigma & 0 \\ \frac{1}{2}e^{-\frac{1}{2}\sigma}F_\sigma(\mathrm{grad}^g(\sigma)\cdot) & e^{-\frac{1}{2}\sigma}F_\sigma \end{pmatrix} \begin{pmatrix} NW(g) & NE(g) \\ SW(g) & SE(g) \end{pmatrix}.\end{aligned}$$

Thus, the following four equations arise:

$$\begin{aligned} NE(\hat{g})(e^{\frac{1-n}{2}\sigma}F_\sigma) &= e^{-\frac{n+1}{2}\sigma}F_\sigma \circ NE(g), \\ NW(\hat{g})(e^{\frac{3-n}{2}\sigma}F_\sigma) + NE(\hat{g})(\frac{1}{2}e^{\frac{1-n}{2}\sigma}F_\sigma(\mathrm{grad}^g(\sigma)\cdot)) &= e^{-\frac{n+1}{2}\sigma}F_\sigma \circ NW(g), \\ SE(\hat{g})(e^{\frac{1-n}{2}\sigma}F_\sigma) &= \frac{1}{2}e^{-\frac{n+3}{2}\sigma}F_\sigma(\,\mathrm{grad}^g(\sigma)\cdot NE(g)) + e^{-\frac{n+3}{2}\sigma}F_\sigma \circ SE, \\ SW(\hat{g})(e^{\frac{3-n}{2}\sigma}F_\sigma) + SE(\hat{g})(\frac{1}{2}e^{\frac{1-n}{2}\sigma}F_\sigma(\mathrm{grad}^g(\sigma)\cdot)) & \\ &= \frac{1}{2}e^{-\frac{n+3}{2}\sigma}F_\sigma(\,\mathrm{grad}^g(\sigma)\cdot NW(g)) + e^{-\frac{n+3}{2}\sigma}F_\sigma \circ SW(g). \end{aligned}$$

The first equation implies that $NE(g)$ should be a conformally covariant differential operator of bi-degree $(\frac{1-n}{2}, -\frac{1+n}{2})$. Since we have assumed the existence of a series of conformally covariant differential operators $D_k(g)$ of bi-degree $(\frac{k-n}{n}, -\frac{k+n}{2})$, we set $NE(g) = D_1(g)$. Hence, the first equation above is fulfilled. The remaining entries can be determined, although not uniquely in any obvious sense, through the equations we are left with.

Remark 5.43 In a joint work with Andreas Juhl, we found, analyzing the operator $\Box^{g,\mathcal{S}(M)}_{\frac{2-n}{2}}$, Proposition 5.28, and its conformal covariance, Proposition 5.26, that the set of equations, given above, is fulfilled by $NE(g) = 2\not{D}$, $NW(g) = SE(g) = -\not{D}^2$ and $SW(g) = (P, \nabla^{S(M,g)}) + \frac{1}{2}\,\mathrm{grad}^g(J)\cdot$. But the operator $SW(g)$ could also be $\frac{1}{2}\not{D}^3$, which is a special case of the next proposition for $D_1 = \not{D}$.

Theorem 5.44 *(Juhl, A. and Fischmann, M.)*
The operator $L_1(g) : \Gamma(\mathcal{S}(M)_g) \to \Gamma(\mathcal{S}(M)_g)$ given by

$$L_1(g) := \begin{pmatrix} -D_1 \circ \not{D} & 2D_1 \\ \frac{1}{2}\not{D} \circ D_1 \circ \not{D} & -\not{D} \circ D_1 \end{pmatrix},$$

satisfies equation (5.16)*, i.e., it is a conformally covariant differential operator with bi-degree $(\frac{2-n}{2}, -\frac{2+n}{2})$.*

Proof. The proof is based on checking the equations given above by hand. This requires the conformal covariance of the Dirac operator $\not{D}$ and of D_1, as well as the product rule for the Dirac operator. Note that the matrix equation gives us four equations. The first one is the conformal covariance of the operator D_1. To prove the second equation, let $\psi \in \Gamma(S(M,g))$ and consider the vector bundle isomorphism $F_\sigma : S(M,g) \to S(M, e^{2\sigma}g)$. Then

$$-\hat{D}_1 \circ \hat{\not{D}}(e^{\frac{3-n}{2}\sigma} F_\sigma \psi) + \hat{D}_1(e^{\frac{1-n}{2}\sigma} F_\sigma(\mathrm{grad}^g(\sigma)\cdot\psi)) = -e^{-\frac{n+1}{2}\sigma} F_\sigma(D_1 \circ \not{D}\psi)$$

is fulfilled, because of

$$\begin{aligned}
-\hat{D}_1 \circ \hat{\not{D}}(e^{\frac{3-n}{2}\sigma} F_\sigma \psi) &= -\hat{D}_1 \circ \hat{\not{D}}(e^{\frac{1-n}{2}\sigma} e^{\sigma} F_\sigma \psi)\\
&= -\hat{D}_1\big[e^{-\frac{n+1}{2}\sigma} F_\sigma(\not{D}(e^{\sigma}\psi))\big]\\
&= -\hat{D}_1\big[e^{\frac{1-n}{2}\sigma}\big(F_\sigma(\mathrm{grad}^g(\sigma)\cdot\psi) + F_\sigma(\not{D}\psi)\big)\big]\\
&= -\hat{D}_1\big(e^{\frac{1-n}{2}\sigma} F_\sigma(\mathrm{grad}^g(\sigma)\cdot\psi)\big) - e^{-\frac{n+1}{2}\sigma} F_\sigma(D_1 \circ \not{D}\psi).
\end{aligned}$$

An analogous computation shows

$$-\hat{\not{D}} \circ \hat{D}_1(e^{\frac{1-n}{2}\sigma} F_\sigma \psi) = e^{-\frac{n+3}{2}\sigma} F_\sigma(\mathrm{grad}^g(\sigma)\cdot D_1\psi) - e^{-\frac{n+3}{2}\sigma} F_\sigma(\not{D} \circ D_1\psi),$$

which is exactly the third equation. To show the fourth equation, we have to compute

$$\begin{aligned}
\frac{1}{2}\hat{\not{D}} \circ \hat{D}_1 \circ \hat{\not{D}}(e^{\frac{3-n}{2}\sigma} F_\sigma \psi) =& \frac{1}{2} e^{-\frac{n+3}{2}\sigma} F_\sigma\Big[\not{D} \circ D_1 \circ \not{D}\psi - \mathrm{grad}^g(\sigma)\cdot D_1 \circ \not{D}\psi\\
&- \mathrm{grad}^g(\sigma)\cdot D_1(\mathrm{grad}^g(\sigma)\cdot\psi) + \not{D} \circ D_1(\mathrm{grad}^g(\sigma)\cdot\psi)\Big]
\end{aligned}$$

and

$$\begin{aligned}
-\frac{1}{2}\hat{\not{D}} \circ \hat{D}_1\big(e^{\frac{1-n}{2}\sigma} F_\sigma(\mathrm{grad}^g(\sigma)\cdot\psi)\big) =& \frac{1}{2} e^{-\frac{n+3}{2}\sigma} F_\sigma\Big(\mathrm{grad}^g(\sigma)\cdot D_1(\mathrm{grad}^g(\sigma)\cdot\psi)\\
&- \not{D} \circ D_1(\mathrm{grad}^g(\sigma)\cdot\psi)\Big).
\end{aligned}$$

This leads to

$$\begin{aligned}
\frac{1}{2}\hat{\not{D}} \circ \hat{D}_1 \circ \hat{\not{D}}(e^{\frac{3-n}{2}\sigma} F_\sigma \psi) - &\frac{1}{2}\hat{\not{D}} \circ \hat{D}_1(e^{\frac{1-n}{2}\sigma} F_\sigma(\mathrm{grad}^g(\sigma)\cdot\psi))\\
&= \frac{1}{2} e^{-\frac{n+3}{2}\sigma} F_\sigma(\not{D} \circ D_1 \circ \not{D}\psi - \mathrm{grad}^g(\sigma)\cdot D_1 \circ \not{D}\psi),
\end{aligned}$$

which is the fourth equation, and the proof is complete. □

Remark 5.45 The essential idea of this construction was to start with a conformally covariant differential operator with the correct bi-degree in the upper right corner, and to guess the missing entries.

Now, we are going to generalize this construction:

Theorem 5.46 *For any odd $k \in \mathbb{N}$ let $D_k(g) : \Gamma(S(M,g)) \to \Gamma(S(M,g))$ be a conformally covariant differential operators of bi-degree $(\frac{k-n}{2}, -\frac{k+n}{2})$. Then the operator $L_k(g) : \Gamma(\mathcal{S}(M)_g) \to \Gamma(\mathcal{S}(M)_g)$ given by*

$$L_k(g) := \begin{pmatrix} -D_k \circ \not{D} & (k+1)D_k \\ \frac{1}{k+1}\not{D} \circ D_k \circ \not{D} & -\not{D} \circ D_k \end{pmatrix},$$

is a conformally covariant differential operator of bi-degree $(\frac{(k+1)-n}{2}, -\frac{(k+1)+n}{2})$.

Proof. Let $g, e^{2\sigma}g \in c$. We use the conformal covariance properties of the Dirac operator $\not{D}$ and $D_k(g)$, and the product rule for the Dirac operator. The equations we have to check come from

$$\begin{aligned}
&\begin{pmatrix} -\hat{D}_k \circ \hat{\not{D}} & (k+1)\hat{D}_k \\ \frac{1}{k+1}\hat{\not{D}} \circ \hat{D}_k \circ \hat{\not{D}} & -\hat{\not{D}} \circ \hat{D}_k \end{pmatrix} e^{\frac{(k+1)-n}{2}\sigma} \begin{pmatrix} e^{\frac{1}{2}\sigma}F_\sigma & 0 \\ \frac{1}{2}e^{-\frac{1}{2}\sigma}F_\sigma(\mathrm{grad}^g(\sigma)\cdot) & e^{-\frac{1}{2}\sigma}F_\sigma \end{pmatrix} \\
=& e^{-\frac{(k+1)+n}{2}\sigma} \begin{pmatrix} e^{\frac{1}{2}\sigma}F_\sigma & 0 \\ \frac{1}{2}e^{-\frac{1}{2}\sigma}F_\sigma(\mathrm{grad}^g(\sigma)\cdot) & e^{-\frac{1}{2}\sigma}F_\sigma \end{pmatrix} \begin{pmatrix} -D_k \circ \not{D} & (k+1)D_k \\ \frac{1}{k+1}\not{D} \circ D_k \circ \not{D} & -\not{D} \circ D_k \end{pmatrix}.
\end{aligned}$$

The first equation is the conformal covariance of the operator D_k. Computing for $\psi \in \Gamma(S(M,g))$,

$$\begin{aligned}
-\hat{D}_k \circ \hat{\not{D}}(e^{\frac{k+2-n}{2}\sigma}F_\sigma\psi) &= -e^{-\frac{n+k}{2}\sigma}F_\sigma\Big(\frac{k+1}{2}D_k(\mathrm{grad}^g(\sigma)\cdot\psi) + D_k \circ \not{D}\psi\Big), \\
\frac{k+1}{2}\hat{D}_k\big(e^{\frac{k-n}{2}\sigma}F_\sigma(\mathrm{grad}^g(\sigma)\cdot\psi)\big) &= \frac{k+1}{2}e^{\frac{n+k}{2}\sigma}F_\sigma\big(D_k(\mathrm{grad}^g(\sigma)\cdot\psi)\big),
\end{aligned}$$

we get the second equation

$$\begin{aligned}
-\hat{D}_k \circ \hat{\not{D}}(e^{\frac{k+2-n}{2}\sigma}F_\sigma\psi) + \frac{k+1}{2}\hat{D}_k\big(e^{\frac{k-n}{2}\sigma}F_\sigma(\mathrm{grad}^g(\sigma)\cdot\psi)\big) \\
= -e^{-\frac{n+k}{2}\sigma}F_\sigma\big(D_k \circ \not{D}\psi\big).
\end{aligned}$$

The third one can be seen analogously:

$$-\hat{\not{D}} \circ \hat{D}_k(e^{\frac{k-n}{2}\sigma}F_\sigma\psi) = \frac{k+1}{2}e^{-\frac{n+k+2}{2}\sigma}F_\sigma\big(\mathrm{grad}^g(\sigma)\cdot D_k\psi\big) - e^{-\frac{n+k+2}{2}\sigma}F_\sigma\big(\not{D} \circ D_k\psi\big).$$

In order to derive the last equation, we compute

$$\begin{aligned}
\frac{1}{k+1}\hat{\not{D}} \circ \hat{D}_k \circ \hat{\not{D}}(e^{\frac{(k+2)-n}{2}\sigma}F_\sigma\psi) &= \frac{1}{k+1}e^{-\frac{n+(k+2)}{2}\sigma}F_\sigma(\not{D} \circ D_k \circ \not{D}\psi) \\
&- \frac{1}{2}e^{-\frac{n+(k+2)}{2}\sigma}F_\sigma(\mathrm{grad}^g(\sigma)\cdot D_k \circ \not{D}\psi) + \frac{1}{2}e^{-\frac{n+(k+2)}{2}\sigma}F_\sigma\big(\not{D} \circ D_k(\mathrm{grad}^g(\sigma)\cdot\psi)\big) \\
&- \frac{k+1}{4}e^{-\frac{n+(k+2)}{2}\sigma}F_\sigma\big(\mathrm{grad}^g(\sigma)\cdot D_k(\mathrm{grad}^g(\sigma)\cdot\psi)\big)
\end{aligned}$$

and

$$\begin{aligned}-\frac{1}{2}\hat{\not{D}}\circ\hat{D}_k(e^{\frac{k-n}{2}\sigma}F_\sigma(\mathrm{grad}^g(\sigma)\cdot\psi)) =&\frac{k+1}{4}e^{-\frac{n+(k+2)}{2}\sigma}F_\sigma\big(\mathrm{grad}^g(\sigma)\cdot D_k(\mathrm{grad}^g(\sigma)\cdot\psi)\big)\\ &-\frac{1}{2}e^{-\frac{n+(k+2)}{2}\sigma}F_\sigma\big(\not{D}\circ D_k(\mathrm{grad}^g(\sigma)\cdot\psi)\big),\end{aligned}$$

hence,

$$\begin{aligned}&\frac{1}{k+1}\hat{\not{D}}\circ\hat{D}_k\circ\hat{\not{D}}(e^{\frac{(k+2)-n}{2}\sigma}F_\sigma\psi)-\frac{1}{2}\hat{\not{D}}\circ\hat{D}_k(e^{\frac{k-n}{2}\sigma}F_\sigma(\mathrm{grad}^g(\sigma)\cdot\psi))\\ &\quad=\frac{1}{k+1}e^{-\frac{n+(k+2)}{2}\sigma}F_\sigma(\not{D}\circ D_k\circ\not{D}\psi)-\frac{1}{2}e^{-\frac{n+(k+2)}{2}\sigma}F_\sigma(\mathrm{grad}^g(\sigma)\cdot D_k\circ\not{D}\psi),\end{aligned}$$

which completes the proof. □

Remark 5.47 The above constructed operators can be represented in another form: Let $k\in\mathbb{N}$ be odd. Then one has

$$\begin{aligned}L_k(g)=&\begin{pmatrix}\sqrt{k+1}\\ -\frac{1}{\sqrt{k+1}}\not{D}\end{pmatrix}D_k\begin{pmatrix}-\frac{1}{\sqrt{k+1}}\not{D} & \sqrt{k+1}\end{pmatrix}\\ =&\frac{4}{k+1}D^{S(M,g)}(g,-\frac{n+k}{2})\circ D_k\circ C^{S(M,g)}(g,\frac{k+1-n}{2}).\end{aligned}$$

This structure of the $L_k(g)$'s ensures their conformal covariance, since their building blocks are conformally covariant differential operators, see Lemma 5.14 and Lemma 5.15.

In what will follow, we will consider conformal powers of the Dirac operator $\mathcal{D}_k$, $k\in\mathbb{N}$ odd, and we will associate to them the corresponding conformally covariant differential operators of bi-degree $(\frac{k+1-n}{2},-\frac{k+1+n}{2})$ acting on the standard spin tractor bundle. These operators will be denoted by $\mathcal{L}_k(g)$. Recall, that we have constructed a series of conformally covariant differential operators $P^{\mathcal{S}(M)}_{k+1}(g)$ of bi-degree $(\frac{k+1-n}{2},-\frac{k+1+n}{2})$ in Section 5.4. The operators $\mathcal{L}_k(g)$ and $P^{\mathcal{S}(M)}_{k+1}(g)$ act on the $g-$trivialized standard spin tractor bundle and are conformally covariant of bi-degree $(\frac{k+1-n}{2},-\frac{k+1+n}{2})$. Although their constructions, equation (5.7) and Remark 5.47, are different, they show some similarities for $k=1,3$. For $k=1$, recall the $g-$metric representation of the operator $P^{\mathcal{S}(M)}_2(g)=\Box^{g,\mathcal{S}(M)}_{\frac{2-n}{2}}$:

$$\Box^{g,\mathcal{S}(M)}_{\frac{2-n}{2}}=\begin{pmatrix}-\not{D}^2 & 2\not{D}\\ (P,\nabla^{S(M,g)})+\frac{1}{2}\mathrm{grad}^g(J)\cdot & -\not{D}^2\end{pmatrix}.$$

Then we have:

Theorem 5.48 *The difference of $\mathcal{L}_1(g)$ and $\Box^{g,\mathcal{S}}_{\frac{2-n}{2}}$ induces a third order conformally* *covariant differential operator of bi-degree* $(\frac{3-n}{2}, -\frac{3+n}{2})$ *acting on spinor fields, and it is given by*

$$\frac{1}{2}\not{D}^3 - P(s_i)^\natural \cdot \nabla^{S(M,g)}_{s_i} - \frac{1}{2}\,\mathrm{grad}^g(J)\cdot,$$

which is equal to $\frac{1}{2}\mathcal{D}_3$.

Proof. The difference produces a matrix with three vanishing entries and a non-vanishing entry in the down left corner. The transformation laws for conformally related metrics of the two ingredients imply a conformal transformation law for the non-vanishing entry with conformal weight $(\frac{3-n}{2}, -\frac{3+n}{2})$. $\square$

Now lets do the same with $\mathcal{L}_3(g)$ and $P_4(g)$, the main part of $P_4^{\mathcal{S}(M)}(g)$. Due to the explicit formula

$$P_4(g) = \begin{pmatrix} (n-4)\mathcal{D}_3 \circ \not{D} & -4(n-4)\mathcal{D}_3 \\ \begin{array}{c} (n-4)\big[\frac{1}{2}\mathcal{D}_3\not{D}^2 + \frac{1}{2}\not{D}^2\mathcal{D}_3 - \not{D}^5 \\ +2\big((P^2, \nabla^{S(M,g)}) + (\nabla^{S(M,g)}, P^2\cdot)\big) \\ +\frac{2}{n-4}(B, \nabla^{S(M,g)}) - (C, P)\cdot\,\big] \end{array} & (n-4)\not{D} \circ \mathcal{D}_3 \end{pmatrix},$$

we can compare it with

$$\mathcal{L}_3(g) = \begin{pmatrix} -\mathcal{D}_3 \circ \not{D} & 4\mathcal{D}_3 \\ \frac{1}{4}\not{D} \circ \mathcal{D}_3 \circ \not{D} & -\not{D} \circ \mathcal{D}_3 \end{pmatrix}$$

to find:

Theorem 5.49 *Let* $n \neq 4$. *The difference of* $\frac{4}{n-4}P_4(g)$ *and* $-4\mathcal{L}_3(g)$ *induces a conformally covariant differential operator of fifth order with bi-degree* $(\frac{5-n}{2}, -\frac{n+5}{2})$, *acting on spinor fields, and it is given by*

$$\not{D}\mathcal{D}_3\not{D} + 2\big[\mathcal{D}_3\not{D}^2 + \not{D}^2\mathcal{D}_3\big] - 4\not{D}^5 + 8\big[(P^2, \nabla^{S(M,g)}) + (\nabla^{S(M,g)}, P^2\cdot)\big]$$
$$+ \frac{4}{n-4}\big[(B, \nabla^{S(M,g)}) + (\nabla^{S(M,g)}, B\cdot)\big] - 2\big[(C, P\cdot) + (P, C\cdot)\big],$$

which is precisely the operator $\frac{1}{n-4}D_5^{red}(g)$, *given in Theorem 5.39.*

Proof. It is obvious that the difference produces a matrix with three vanishing entries and a non-vanishing entry in the down left corner. The conformal covariance property with the mentioned bi-degree follows from the transformation laws for conformally related metrics of the two ingredients . $\square$

Remark 5.50 Recall the decomposition of the operator $P_4^{\mathcal{S}(M)}(g) = P_4(g) + R(g)$, equation (5.15), into two conformally covariant differential operators. If one demands that $\frac{4}{n-4}P_4^{\mathcal{S}(M)}(g)+4\mathcal{L}_3(g)$ induces, directly, a conformally covariant differential operator on the spinor bundle, the operator $\mathcal{L}_3(g)$ indicates such a decomposition of $P_4^{\mathcal{S}(M)}(g)$.

6 Further structures of conformal powers of the Dirac operator

In this chapter we will present a new structure of the first examples of conformal powers of the Dirac operator. This is closely related to Juhl's inversion formulas [Juh11] for conformal powers of the Laplacian.

We have presented explicit formulas for conformal powers of the Dirac operator of order three and five acting on the spinor bundle. Let us recall these formulas:

$$\begin{aligned}
\mathcal{D}_1 =& \not{D},\\
\mathcal{D}_3 =& \mathcal{D}_1^3 - (P, \nabla^{S(M,g)}) - (\nabla^{S(M,g)}, P\cdot),\\
\mathcal{D}_5 =& \mathcal{D}_1\mathcal{D}_3\mathcal{D}_1 + 2(\mathcal{D}_1^2\mathcal{D}_3 + \mathcal{D}_3\mathcal{D}_1^2) - 4\mathcal{D}_1^5 + 8(P^2, \nabla^{S(M,g)}) + 8(\nabla^{S(M,g)}, P^2\cdot)\\
&+ \frac{4}{n-4}\left[(B, \nabla^{S(M,g)}) + (\nabla^{S(M,g)}, B\cdot)\right] - 2\left[(C, P\cdot) + (P, C\cdot)\right].
\end{aligned}$$

Here we have used, respectively, the notation from the equations (2.4), (2.6), (5.12), (5.13) and (5.14).

The leading term of these operators is given by $\not{D}^k$ for appropriate odd $k \in \mathbb{N}$. From this we see that the first three examples carry a special structure, namely a composition of lower order $\mathcal{D}_k$'s and a term containing only first and zero derivatives. Let us put the latter ones into a new object, called M_k, $k = 1, 3, 5$. Hence,

$$\begin{aligned}
M_1 :=& \mathcal{D}_1 - 0\\
M_3 :=& \mathcal{D}_3 - \mathcal{D}_1^3\\
M_5 :=& \mathcal{D}_5 - \mathcal{D}_1\mathcal{D}_3\mathcal{D}_1 - 2(\mathcal{D}_1^2\mathcal{D}_3 + \mathcal{D}_3\mathcal{D}_1^2) + 4\mathcal{D}_1^5.
\end{aligned}$$

The zero above just indicates that we can not substract any lower $\mathcal{D}_k$ from $\mathcal{D}_1$. The essential idea is that we can represent the operator $\mathcal{D}_k$ through a composition of M_k's, $k = 1, 3, 5$, and vice versa:

Theorem 6.1 *On a spin manifold (M, g) one has*

$$\begin{aligned}
\mathcal{D}_1 =& M_1,\\
\mathcal{D}_3 =& M_1^3 + M_3,\\
\mathcal{D}_5 =& M_1^5 + M_1M_3M_1 + 2(M_1^2M_3 + M_3M_1^2) + M_5.
\end{aligned}$$

Furthermore, it holds that

$$
\begin{aligned}
M_1 =& \mathcal{D}_1, \\
M_3 =& \mathcal{D}_3 - \mathcal{D}_1^3, \\
M_5 =& \mathcal{D}_5 - \mathcal{D}_1 \mathcal{D}_3 \mathcal{D}_1 - 2(\mathcal{D}_1^2 \mathcal{D}_3 + \mathcal{D}_3 \mathcal{D}^2) + 4\mathcal{D}_1^5.
\end{aligned}
$$

Remark 6.2 We believe, inspired from [Juh11, FG13], that these new first order differential operators M_1, M_3 and M_5 can be derived from a more fundamental object induced by the Poincaré metric, i.e., an appropriately chosen deformation of the Dirac operator induced by the Poincaré metric.

The next proposition presents an additional structure of the operators M_1, M_3 and M_5 similar to [Juh10, Theorem 3.1].

Proposition 6.3 *Let $g_t := e^{2\sigma t} g$ be conformally equivalent to $g \in c$. Then one has*

$$
\begin{aligned}
\frac{d}{dt}_{|t=0}(e^{\frac{n+1}{2}\sigma t} M_1(g_t)(e^{\frac{1-n}{2}\sigma t}\hat{\psi}_t)) =& 0 \\
\frac{d}{dt}_{|t=0}(e^{\frac{n+3}{2}\sigma t} M_3(g_t)(e^{\frac{3-n}{2}\sigma t}\hat{\psi}_t)) =& -\big[M_1, [M_1^2, \sigma]\big], \\
\frac{d}{dt}_{|t=0}(e^{\frac{n+5}{2}\sigma t} M_5(g_t)(e^{\frac{5-n}{2}\sigma t}\hat{\psi}_t)) =& -2\big[M_1, [M_1 M_3 + M_3 M_1, \sigma]\big] - 4\big[M_3, [M_1^2, \sigma]\big]
\end{aligned}
$$

where $\hat{\psi}_t := F_{t\sigma}(\psi)$ for some $\psi \in \Gamma(S(M, g))$.

Proof. The first identity is just the conformal covariance of the Dirac operator in its infinitesimal form. To derive the second equation, we compute

$$
\begin{aligned}
\frac{d}{dt}_{|t=0}(e^{\frac{n+3}{2}\sigma t} M_3(g_t)(e^{\frac{3-n}{2}\sigma t}\hat{\psi}_t)) =& -\frac{d}{dt}_{|t=0}\Big(F_{\sigma t}(e^{\sigma t}\not{D} e^{-\sigma t}\not{D} e^{-\sigma t}\not{D} e^{\sigma t}\psi)\Big) \\
=& -\big[\not{D}^2, [\not{D}, \sigma]\big] = -[\not{D}, [\not{D}^2, \sigma]] \\
=& -\big[M_1, [M_1^2, \sigma]\big].
\end{aligned}
$$

The last equation follows from

$$
\begin{aligned}
\frac{d}{dt}_{|t=0}(e^{\frac{n+5}{2}\sigma t}\not{D}(g_t)\mathcal{D}_3(g_t)\not{D}(g_t)(e^{\frac{5-n}{2}\sigma t}\hat{\psi}_t)) =& 2M_1\mathcal{D}_3[M_1, \sigma] - 2[M_1, \sigma]\mathcal{D}_3 M_1, \\
\frac{d}{dt}_{|t=0}(e^{\frac{n+5}{2}\sigma t}\mathcal{D}_3(g_t)\not{D}^2(g_t)(e^{\frac{5-n}{2}\sigma t}\hat{\psi}_t)) =& -[\mathcal{D}_3, \sigma]M_1^2 + \mathcal{D}_3[M_1^2, \sigma] + \mathcal{D}_3 M_1[M_1, \sigma], \\
\frac{d}{dt}_{|t=0}(e^{\frac{n+5}{2}\sigma t}\not{D}^2(g_t)\mathcal{D}_3(g_t)(e^{\frac{5-n}{2}\sigma t}\hat{\psi}_t)) =& M_1^2[\mathcal{D}_3, \sigma] - [M_1^2, \sigma]\mathcal{D}_3
\end{aligned}
$$

$$
\begin{aligned}
&- [M_1, \sigma] M_1 \mathcal{D}_3, \\
\frac{d}{dt}_{|t=0}(e^{\frac{n+5}{2}\sigma t} D\!\!\!/^5(g_t)(e^{\frac{5-n}{2}\sigma t}\hat{\psi}_t)) =& [M_1^4, [M_1, \sigma]] + [M_1^3, [M_1^2, \sigma]],
\end{aligned}
$$

and $\mathcal{D}_3 = M_1^3 + M_3$, hence,

$$
\begin{aligned}
\frac{d}{dt}_{|t=0}(e^{\frac{n+5}{2}\sigma t} M_5(g_t)(e^{\frac{5-n}{2}\sigma t}\hat{\psi}_t)) =& - 2\big[M_1, [M_1 M_3, \sigma]\big] - 2\big[M_1, [M_3 M_1, \sigma]\big] \\
&- 4\big[M_3, [M_1^2, \sigma]\big],
\end{aligned}
$$

which proves the proposition. □

Remark 6.4 The construction of M_k, $k = 1, 3, 5$, starts with a combination of conformally covariant operators in such way that the order of each summand equals k. The weights in front of the combinations of lower $\mathcal{D}_j$ we have chosen come from the explicit form of the operators $\mathcal{D}_k$, $k = 1, 3, 5$. A universal formula for the weights (without knowing the operator) is not clear yet, but there is some hope, that one can prove a formula for the M_k in terms of $\mathcal{D}_l$ for $l \leq k$ odd. Another phenomenon is that the lower examples of $\mathcal{D}_k$ have a representation in terms of M_l for $l \leq k$. Again, we hope to find an universal formula for $\mathcal{D}_k$ in terms of M_l for $l \leq k$ odd. This is inspired by the inversion formula, found in [Juh11, Section 2] for the conformal powers of the Laplacian.

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 18.10.2012 Matthias Fischmann